

Algebraic representations of von Neumann algebras

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Abstract

An (algebraic) extended bilinear Hilbert semispace H_a^\mp is proposed as being the natural representation space for the algebras of von Neumann. This bilinear Hilbert semispace has a well defined structure given by the representation space $\text{Repsp}(GL_n(L_{\bar{v}} \times L_v))$ of an (algebraic) complete bilinear semigroup $GL_n(L_{\bar{v}} \times L_v)$ over the product of sets of completions characterized by increasing ranks.

This representation space is a $GL_n(L_{\bar{v}}^{(nr)} \times L_v^{(nr)})$ -bisemimodule $M_R^{(nr)} \otimes M_L^{(nr)}$, decomposing into subbisemimodules according to the pseudounramified or pseudo-ramified conjugacy classes of $GL_n(L_{\bar{v}}^{(nr)} \times L_v^{(nr)})$, and is in one-to-one correspondence with its cuspidal representation according to the Langlands global program.

In this context, towers of von Neumann subbisemialgebras on graded bilinear Hilbert subsemispaces, of which structures are these subbisemimodules, are constructed algebraically which allows to envisage the classification of the factors of von Neumann from an algebraic point of view.

Introduction

The first essential step of this paper consists in building up a bilinear mathematical frame for the representations of the von Neumann algebras in such a way that the most convenient representation space be essentially an extended bilinear Hilbert semispace characterized by a non-orthogonal basis.

Considering that the representation space of a von Neumann algebra must be the enveloping algebra [13] of the Hilbert (semi)module on which this von Neumann algebra acts, an extended bilinear Hilbert semispace is then proposed whose Hilbert bisemimodule constitutes the searched enveloping semialgebra [30]: this constitutes the content of chapter 1 [29].

In this perspective, an algebraic (real) extended bilinear Hilbert semispace H_a^\pm and an analytic (complex) extended bilinear Hilbert semispace H_h^\pm are constructed and proved to be the natural representation spaces for the algebras of elliptic operators. In this context, semialgebras and bisemialgebras of von Neumann on the spaces H_a^\pm and H_h^\pm are introduced according to the general treatment of semistructures and bisemistructures introduced in [30].

The generation of algebraic bilinear Hilbert semispaces is related to the bilinear Eisenstein cohomology which constitutes the algebraic pillar of the bilinear global program of Langlands introduced in [29]. More concretely, we are interested in the representation space $\text{Repsp}(GL_n(L_{\bar{v}} \times L_v))$ of a bilinear general semigroup over the product $(L_{\bar{v}} \times L_v)$ of sets of pseudoramified real completions, at infinite archimedean places, whose ranks (or degrees) are integers modulo N in such a way that:

- $GL_n(L_{\bar{v}} \times L_v)$ has the Gauss bilinear decomposition into the product of subgroups of diagonal matrices by the subgroups of upper and lower unitriangular matrices;
- $GL_n(L_{\bar{v}} \times L_v) = T_n^t(L_{\bar{v}}) \times T_n(L_v)$ has for representation space the tensor product $(M_R \otimes M_L)$ of a right $T_n^t(L_{\bar{v}})$ -semimodule M_R by a left $T_n(L_v)$ -semimodule M_L such that M_L (resp. M_R) decomposes into $T_n(L_{v_i})$ -subsemimodules M_{v_i} (resp. $T_n^t(L_{\bar{v}_i})$ -subsemimodules $M_{\bar{v}_i}$) according to the left (resp. right) archimedean places v_i (resp. \bar{v}_i) of L_v (resp. $L_{\bar{v}}$) and so that the set of left (resp. right) subsemimodules M_{v_i} (resp. $M_{\bar{v}_i}$) corresponds to the set of left (resp. right) conjugacy classes of $GL_n(L_{\bar{v}} \times L_v)$.

The bilinear Eisenstein cohomology (semi)group is the cohomology of the Shimura bisemivariety given by

$$\partial\overline{S}_{G_{R \times L}} = P_n(L_{\overline{v}^1} \times L_{v^1}) \setminus GL_n(L_R^+ \times L_L^+) / GL_n((\mathbb{Z} / N \mathbb{Z})^2)$$

where

- $P_n(L_{\overline{v}^1} \times L_{v^1})$ is a bilinear parabolic subsemigroup over the product, right by left, of sets of irreducible real completions having a rank N ;
- $GL_n((\mathbb{Z} / N \mathbb{Z})^2)$ is a bilinear arithmetic subsemigroup constituting the representation of the tensor product of Hecke operators and having a representation in a Hecke bilattice;
- L_R^+ and L_L^+ are symmetric (real) algebraic (semi)fields.

Then, the bilinear Eisenstein cohomology

$$H^{2j}(\partial\overline{S}_{G_{R \times L}}, \widetilde{M}_R^{2j} \otimes \widetilde{M}_L^{2j}) \simeq \text{Repsp}(GL_{2j}(L_{\overline{v}} \times L_v)) , \quad 2j \leq n ,$$

of the Shimura bisemivariety $\partial\overline{S}_{G_{R \times L}}$ has coefficients in the (bisemi)sheaf $\widetilde{M}_R^{2j} \otimes \widetilde{M}_L^{2j}$ over the $GL_{2j}(L_{\overline{v}} \times L_v)$ -bisemimodule $(M_R^{2j} \otimes M_L^{2j})$ and is in bijection with the representation space of the complete bilinear algebraic semigroup $GL_{2j}(L_{\overline{v}} \times L_v)$.

Furthermore, the complete reducibility of $\text{Repsp}(GL_{2n}(L_{\overline{v}} \times L_v))$ induces the decomposition of the bilinear Eisenstein cohomology into (irreducible) two-dimensional bilinear Eisenstein cohomologies.

On the other hand, the analytic pillar of the global program of Langlands is given by the cuspidal representation of the coefficients of the bilinear Eisenstein cohomology in terms of products, right by left, of global elliptic semimodules which are (truncated) Fourier series over \mathbb{R} whose number of terms corresponds to the number of conjugacy classes of the general bilinear semigroup $GL_{2j}(L_{\overline{v}} \times L_v)$.

The Eisenstein and analytic de Rham cohomologies are considered and recalled to be isomorphic [21] from which it results that (bi)semialgebras of von Neumann on the algebraic and analytic bilinear Hilbert semispaces H_a^\pm and H_h^\pm are isomorphic:

$$\mathbb{M}_{\substack{R,L \\ R \times L}}(H_a^\pm) \simeq \mathbb{M}_{\substack{R,L \\ R \times L}}(H_h^\mp) .$$

The action of a (differential) bioperator $(T_R^D \otimes T_L^D) \in \mathbb{M}_{R \times L}(H_a^\mp)$ of rank $(m \times m)$, with $m \leq n$, (associated with a principal $GL_m(\mathbb{R} \times \mathbb{R})$ -bundle) on the $(n \times n)$ -dimensional (bisemi)sheaf $(\widetilde{M}_R \otimes \widetilde{M}_L)$ consists in mapping $(\widetilde{M}_R \otimes \widetilde{M}_L)$ into the corresponding (bisemi)sheaf $(\widetilde{M}_{R_{n[m]}} \otimes \widetilde{M}_{L_{n[m]}})$ shifted into $(m \times m)$ dimensions such that $(\widetilde{M}_{R_{n[m]}} \otimes \widetilde{M}_{L_{n[m]}})$ decomposes into subbisemisheaves according to:

- the pseudoramified conjugacy classes $g_R(i) \times g_L(i)$, $1 \leq i \leq q$, of $GL_n(L_{\overline{v}} \times L_v)$ where “ i ” denotes a global residue degree;

or • according to the pseudounramified conjugacy classes $\gamma_R(i) \times \gamma_L(i)$, $1 \leq i \leq q$, of the pseudounramified bilinear semigroup $GL_n(L_{\overline{v}}^{nr} \times L_v^{nr})$ over sets $L_{\overline{v}}^{nr}$ and L_v^{nr} of pseudounramified completions;

in such a way that:

- $(\widetilde{M}_{R_{n[m]}} \otimes \widetilde{M}_{L_{n[m]}})$ be the coefficient system of the shifted bilinear Eisenstein cohomology $H^{2j-2k}(\partial \overline{S}_{G_{R \times L; n[m]}} , \widetilde{M}_{R_{2j[2k]}}^{2j} \otimes \widetilde{M}_{L_{2j[2k]}}^{2j})$ where

$$\begin{aligned} \partial \overline{S}_{G_{R \times L; n[m]}} = & P_{n[m]}((L_{\overline{v}^1} \otimes \mathbb{R}) \times (L_{v^1} \otimes \mathbb{R})) \\ & \setminus GL_{n[m]}((L_R^+ \otimes \mathbb{R}) \times (L_L^+ \otimes \mathbb{R})) / GL_{n[m]}(\mathbb{Z} / N \mathbb{Z})^2 \otimes \mathbb{R}^2 \end{aligned}$$

is the shifted Shimura bisemivariety;

- $(\widetilde{M}_{R_{2j[2k]}}^{2j} \otimes \widetilde{M}_{L_{2j[2k]}}^{2j})$ decomposes into shifted subbisemisheaves according to the pseudoramified or pseudounramified conjugacy classes of $GL_{2j}(L_{\overline{v}}^{(nr)} \times L_v^{(nr)})$ in such a way that the pseudoramified conjugacy classes correspond to the cosets of $GL_{n[m]}((L_R^+ \otimes \mathbb{R}) \times (L_L^+ \otimes \mathbb{R})) / GL_{n[m]}(\mathbb{Z} / N \mathbb{Z})^2 \otimes \mathbb{R}^2$.

As in the unshifted case, the shifted bilinear Eisenstein cohomology decomposes into direct sum of completely irreducible orthogonal or nonorthogonal shifted bilinear Eisenstein cohomologies.

Taking into account the decomposition of the complete algebraic (resp. analytic) $GL_n(L_{\overline{v}}^{(nr)} \times L_v^{(nr)})$ -bisemimodule $(M_R^{(nr)} \otimes M_L^{(nr)})$ (resp. $(M_R^{s,(nr)} \otimes M_L^{s,(nr)})$) into subbisemimodules according to its pseudounramified or pseudoramified conjugacy classes, the complete algebraic (resp. analytic) extended bilinear Hilbert semispace $H_a^{\mp,(nr)}$ (resp. $H_h^{\mp,(nr)}$)

also decomposes into bilinear subsemispaces $H_a^{\mp, (nr)}(i)$ (resp. $H_h^{\mp, (nr)}(i)$), $1 \leq i \leq q$, or according to sums of bilinear subsemispaces:

$$H_a^{\mp, nr}\{i\} = \bigoplus_{\nu=1}^i H_a^{\mp, nr}(\nu) \quad (\text{resp.} \quad H_h^{\mp, nr}\{i\} = \bigoplus_{\nu=1}^i H_h^{\mp, nr}(\nu)), \quad 1 \leq i \leq q,$$

$$\text{or} \quad H_a^{\mp}\{i\} = \bigoplus_{j=1}^i H_a^{\mp}(j) \quad (\text{resp.} \quad H_h^{\mp}\{i\} = \bigoplus_{j=1}^i H_h^{\mp}(j)).$$

So, towers of sums of embedded bilinear Hilbert subsemispaces

$$\begin{aligned} H_a^{\mp, nr}\{1\} &\subset \cdots \subset H_a^{\mp, nr}\{i\} \subset \cdots \subset H_a^{\mp, nr}\{q\}, \\ H_a^{\mp}\{1\} &\subset \cdots \subset H_a^{\mp}\{i\} \subset \cdots \subset H_a^{\mp}\{q\}, \\ (\text{resp.} \quad H_h^{\mp, nr}\{1\} &\subset \cdots \subset H_h^{\mp, nr}\{i\} \subset \cdots \subset H_h^{\mp, nr}\{q\}, \\ H_h^{\mp}\{1\} &\subset \cdots \subset H_h^{\mp}\{i\} \subset \cdots \subset H_h^{\mp}\{q\}), \end{aligned}$$

can be constructed, leading to consider that these bilinear Hilbert semispaces are “solvable” and graded.

And towers of sums of pseudounramified or pseudoramified von Neumann sub(bi)-semialgebras can be generated according to:

$$\mathbb{M}_{\left\{ \begin{smallmatrix} R, L \\ R \times L \end{smallmatrix} \right\}}(H_a^{\mp, (nr)}\{1\}) \subset \cdots \subset \mathbb{M}_{\left\{ \begin{smallmatrix} R, L \\ R \times L \end{smallmatrix} \right\}}(H_a^{\mp, (nr)}\{i\}) \subset \cdots \subset \mathbb{M}_{\left\{ \begin{smallmatrix} R, L \\ R \times L \end{smallmatrix} \right\}}(H_a^{\mp, (nr)}\{q\}).$$

Then, the discrete spectrum $\sigma(T_R^D \otimes T_L^D)$ of a (differential) bioperator $(T_R^D \otimes T_L^D) \in \mathbb{M}_{R \times L}(H_a^{\mp, (nr)})$ is obtained throughout the morphism from the von Neumann bisemialgebra $\mathbb{M}_{R \times L}(H_a^{\mp, (nr)})$ to the set of von Neumann subbisemialgebras $[\mathbb{M}_{R \times L}(\mathcal{H}_a^{\mp, (nr)}\{i\})]_i$ defined on the set of pseudounramified or pseudoramified bilinear Hilbert subsemispaces $\mathcal{H}_a^{\mp, (nr)}\{i\}$ characterized by a diagonal metric associated with an orthonormal bilinear basis.

If the cuspidal representation space of the $GL_n(L_v^{(nr)} \times L_v^{(nr)})$ -bisemimodule $(M_R^{(nr)} \otimes M_L^{(nr)})$ is taken into account, the corresponding set of eigenbifunctions of the differential bioperator $(T_R^D \otimes T_L^D)$ is given, according to the Langlands program, by the global elliptic subbisemimodules which are products, right by left, of (truncated) Fourier series (over \mathbb{R}) whose number of terms correspond to the number of archimedean places associated with the considered algebraic intermediate finite number (semi)fields.

In this context, the classification of the factors of von Neumann can be envisaged from the algebraic frame developed in this paper.

In correspondence with the introduction of **pseudoramified** bilinear Hilbert semispaces H_a^\pm and of towers of embedded bilinear Hilbert subsemispaces, **pseudounramified** bilinear Hilbert semispaces H_a^{nr} can be defined as well as towers of embedded bilinear pseudounramified Hilbert subsemispaces.

So, if “ i ” labels an algebraic intermediate (semi)field or the associated archimedean completion, $\mathbb{M}_{R,L}(H_a^{\text{nr}}(i))$ will refer to a factor of type I_i while if “ j ” denotes an algebraic internal dimension, $\mathbb{M}_{R,L}(H_a^{\mp,\text{in}}(j))$, $1 \leq j \leq N$, will be a hyperfinite subfactor of type II_{1_j} [23], [24], where N is the order of a global inertia subgroup.

So, our main proposition can finally be stated as follows :

1. On the pseudounramified bilinear Hilbert semispace H_a^{nr} , there are q factors of type I_i , $1 \leq i \leq q \leq \infty$ where “ i ” denotes a global residue degree.
2. On the bilinear Hilbert subsemispace $H_a^{\text{in}}[L_{\bar{v}_1} \times L_{v_1}]$ restricted to the representation of the bilinear parabolic subsemigroup $P_n(L_{\bar{v}_1} \times L_{v_1})$, there are N subfactors of type II_{1_j} , where j denotes an internal algebraic dimension.

The upper subfactor II_{1_N} is the hyperfinite factor II_1 .

3. On the tensor products $H_a^{\text{nr}}(i) \otimes H_a^{\text{in}}(N)$, there are q pseudoramified factors of type II_∞ , $1 \leq i \leq q \leq \infty$, noted $\mathbb{M}_{R,L}(H_a^{\text{nr}}(i) \otimes H_a^{\text{in}}(N))$ where i denotes a global residue degree.
4. On the tensor products $H_a^{\text{nr}}(\infty) \otimes H_a^\mp(i)$, $1 \leq i \leq N$, the factors of type II_∞ are defined.

1 Bilinear semigroups and bilinear Hilbert semispaces

The aim of this chapter is to introduce a sufficiently general mathematical frame for the representations of the von Neumann algebras. As the “representation” of a k -algebra M_L over a number field k of characteristic zero proceeds from its enveloping algebra, the most natural representation space for the von Neumann algebras will be an extended Hilbert semispace of bilinear nature which must then correspond to the representation space of the k -algebra M_L in a linear Hilbert space \mathcal{H} .

If the representation space of a von Neumann algebra is assumed to be non commutative, its (algebraic)-geometric structure will then be of Riemann type and composed of the tensor product of a pair of faithfully projective isomorphic k -semimodules leading to an extended bilinear Hilbert semispace by projection of one of these semimodules on its copy.

Notations: R, L means “ R ” or “ L ” for “right” or “left”;

$\times_{(D)}$ means a diagonal (\times_D) or complete (\times) product.

Definition 1.1 Enveloping algebra: Let $M_{R,L}$ be a k -algebra considered as a finitely generated, projective and faithful right (resp. left) k -module. Its enveloping algebra is given by $M^e = M_R \otimes_k M_L$ where M_R (resp. M_L) is a right (resp. left) k -module viewed as the opposite algebra of M_L (resp. M_R) [13]. If the homomorphism $E_{h_{R,L}} : M^e \rightarrow \text{End}_k(M_{R,L})$ is an isomorphism, then the k -algebra $M_{R,L}$ is called an Azumaya algebra.

If $M_{R,L}$ is a faithfully projective right (resp. left) k -module of dimension n , then $M_{R,L} \simeq k^n$ and we have that [6], [18], [37]:

$$M^e \simeq \text{End}_k(M_{R,L}) \simeq \text{End}_k(k^n) \simeq M_n(k)$$

where $M_n(k)$ is the ring of matrices of order n over k .

The homomorphism $E_{R,L} : M_{R,L} \rightarrow M_n(k)$ is called a n -dimensional representation of $M_{R,L}$ [2].

Definition 1.2 Symmetric algebraic extension field: Let k be a number field of characteristic 0 and L^+ (resp. L) denote a finite real (resp. complex) extension of k . A real (resp. complex) algebraic extension field L^+ (resp. L) will be said symmetric if it is composed of the set of positive (resp. complex) simple roots, noted L_L^+ (resp. L_L), in one-to-one correspondence with the set of negative (resp. complex conjugate) simple roots, noted L_R^+ (resp. L_R), such that to each positive (resp. complex) simple root $x_L^+ \in L_L^+$ (resp. $x_L \in L_L$) corresponds a symmetric negative (resp. complex conjugate) simple root $x_R^+ \in L_R^+$ (resp. $x_R \in L_R$). Geometrically, L_L is then localized in the upper halfspace and L_R in the lower half space. L_L^+ (resp. L_L) and L_R^+ (resp. L_R) are then respectively left and right semifields, i.e. commutative division left and right semirings.

L_L and L_R are semirings because they are abelian semigroups with respect to the addition and are endowed with associative multiplication and distributive laws.

Definition 1.3 Completions associated with finite algebraic extensions: The equivalence classes of the real completions of L_L^+ (resp. L_R^+), obtained by an isomorphism of compactification of the corresponding extensions, are the left (resp. right) infinite places of L_L^+ (resp. L_R^+) and are noted $v = \{v_1, \dots, v_i, \dots, v_q\}$ (resp. $\bar{v} = \{\bar{v}_1, \dots, \bar{v}_i, \dots, \bar{v}_q\}$).

Similarly, the equivalence classes of the complex completions of L_L (resp. L_R), obtained by an isomorphism of compactification of corresponding finite extensions, are the left (resp.

right) infinite complex places of L_L (resp. L_R) and are noted $\omega = \{\omega_1, \dots, \omega_i, \dots, \omega_q\}$ (resp. $\bar{\omega} = \{\bar{\omega}_1, \dots, \bar{\omega}_i, \dots, \bar{\omega}_q\}$).

Let L_{v_i} (resp. $L_{\bar{v}_i}$) denote i -th basic real completion corresponding to the i -th left (resp. right) pseudoramified algebraic extension $L_{L_i}^+$ (resp. $L_{R_i}^+$) of k and associated to the left (resp. right) place v_i (resp. \bar{v}_i). The other equivalent completions of v_i (resp. \bar{v}_i) are noted L_{v_i, m_i} (resp. $L_{\bar{v}_i, m_i}$), where $m_i \in \mathbb{N}$, $m_i > 0$, are increasing integers. ($m_i = 0$ refers to the basic completion L_{v_i} (resp. $L_{\bar{v}_i}$)).

It is assumed that the left (resp. right) pseudoramified completions L_{v_i, m_i} (resp. $L_{\bar{v}_i, m_i}$) are generated from an irreducible completion $L_{v_i^1}$ (resp. $L_{\bar{v}_i^1}$) having a rank or degree equal to N .

Then, the rank of the pseudoramified completions L_{v_i} (resp. $L_{\bar{v}_i}$) and L_{v_i, m_i} (resp. $L_{\bar{v}_i, m_i}$), corresponding to the degree of extension of the associated extension, is given by an integer modulo N according to:

$$\begin{aligned} n_{i_L} &= [L_{v_i, m_i} : k] = * + f_{v_i} \cdot N \simeq i \cdot N \\ (\text{resp. } n_{i_R} &= [L_{\bar{v}_i, m_i} : k] = * + f_{\bar{v}_i} \cdot N \simeq i \cdot N) \end{aligned}$$

where

- $*$ denotes an integer inferior to N ;
- f_{v_i} (resp. $f_{\bar{v}_i}$), called a global class residue degree, is the degree of the corresponding pseudounramified completions L_{v_i, m_i}^{nr} (resp. $L_{\bar{v}_i, m_i}^{nr}$) given by

$$[L_{v_i, m_i}^{nr} : k] = f_{v_i} = i \quad (\text{resp. } [L_{\bar{v}_i, m_i}^{nr} : k] = f_{\bar{v}_i} = i)$$

So, the ranks or degrees of the pseudoramified completions L_{v_i, m_i} (resp. $L_{\bar{v}_i, m_i}$), $1 \leq i \leq q$, are integers modulo N , $\mathbb{Z}/N\mathbb{Z}$.

Remark that the integer $\text{sup}(m_i)$ is interpreted as the multiplicity of the place v_i (resp. \bar{v}_i).

As the rank n_{i_L} (resp. n_{i_R}) of the completion L_{v_i, m_i} (resp. $L_{\bar{v}_i, m_i}$) is assumed to be a multiple of the integer N , which is the rank of the irreducible subcompletion $L_{v_i^1}$ (resp. $L_{\bar{v}_i^1}$), the completion L_{v_i, m_i} (resp. $L_{\bar{v}_i, m_i}$) will be cut into a set of i equivalent subcompletions $L_{v_i^{i'}, m_i}$ (resp. $L_{\bar{v}_i^{i'}, m_i}$), $1 \leq i' \leq i$, of rank N .

Finally, let

$$\begin{aligned} L_v &= \{L_{v_1}, \dots, L_{v_i, m_i}, \dots, L_{v_q, m_q}\} \\ (\text{resp. } L_{\bar{v}} &= \{L_{\bar{v}_1}, \dots, L_{\bar{v}_i, m_i}, \dots, L_{\bar{v}_q, m_q}\}) \end{aligned}$$

denote the set of real pseudoramified completions of L_L^+ (resp. L_R^+) with

$$L_{v_\oplus} = \bigoplus_i \bigoplus_{m_i} L_{v_i, m_i} \quad (\text{resp. } L_{\bar{v}_\oplus} = \bigoplus_i \bigoplus_{m_i} L_{\bar{v}_i, m_i})$$

be their direct sum and let

$$L_v^{nr} = \{L_{v_1}^{nr}, \dots, L_{v_i, m_i}^{nr}, \dots, L_{v_q, m_q}^{nr}\} \\ (\text{resp. } L_{\bar{v}}^{nr} = \{L_{\bar{v}_1}^{nr}, \dots, L_{\bar{v}_i, m_i}^{nr}, \dots, L_{\bar{v}_q, m_q}^{nr}\})$$

denote the corresponding set of real pseudounramified completions.

Similarly, let

$$L_\omega = \{L_{\omega_1}, \dots, L_{\omega_i, m_i}, \dots, L_{\omega_q, m_q}\} \\ (\text{resp. } L_{\bar{\omega}} = \{L_{\bar{\omega}_1}, \dots, L_{\bar{\omega}_i, m_i}, \dots, L_{\bar{\omega}_q, m_q}\})$$

denote the set of complex pseudoramified completions of L_L (resp. L_R) in such a way that the set L_v (resp. $L_{\bar{v}}$) of real completions covers the corresponding set L_ω (resp. $L_{\bar{\omega}}$) of complex completions [29].

Definition 1.4 Galois subgroups and inertia subgroups: Let $\text{Gal}(L_{L_i}^+/k)$ (resp. $\text{Gal}(L_{R_i}^+/k)$) be the Galois subgroup of the pseudoramified extension $L_{L_i}^+$ (resp. $L_{R_i}^+$) and let $\text{Gal}(L_{L_i}^{nr,+}/k)$ (resp. $\text{Gal}(L_{R_i}^{nr,+}/k)$) denote the Galois subgroup of the corresponding pseudounramified extension $L_{L_i}^{nr,+}$ (resp. $L_{R_i}^{nr,+}$).

If $I_{L_{L_i}^+}$ (resp. $I_{L_{R_i}^+}$), denoting the global inertia subgroup of $\text{Gal}(L_{L_i}^+/k)$ (resp. $\text{Gal}(L_{R_i}^+/k)$), is the group of Galois automorphisms of the irreducible extension $L_{L_i}^+$ (resp. $L_{R_i}^+$) or the group of Galois inner automorphisms, then we have that

$$\text{Gal}(L_{L_i}^+/k)/I_{L_{L_i}^+} = \text{Gal}(L_{L_i}^{nr,+}/k) \\ (\text{resp. } \text{Gal}(L_{R_i}^+/k)/I_{L_{R_i}^+} = \text{Gal}(L_{R_i}^{nr,+}/k))$$

such that the exact sequence:

$$0 \longrightarrow I_{L_{L_i}^+} \longrightarrow \text{Gal}(L_{L_i}^+/k) \longrightarrow \text{Gal}(L_{L_i}^{nr,+}/k) \longrightarrow 1 \\ (\text{resp. } 0 \longrightarrow I_{L_{R_i}^+} \longrightarrow \text{Gal}(L_{R_i}^+/k) \longrightarrow \text{Gal}(L_{R_i}^{nr,+}/k) \longrightarrow 1)$$

has kernel given by the global inertia subgroup $I_{L_{L_i}^+}$ (resp. $I_{L_{R_i}^+}$) associated to the place v_i (resp. \bar{v}_i).

If m_i refers to the multiplicity of the left and right places v_i and \bar{v}_i , then the left (resp. right) Galois group can be decomposed according to:

$$\text{Gal}(L_L^+/k) = \bigoplus_{i=1}^q \bigoplus_{m_i} \text{Gal}(L_{L_i, m_i}^+/k)$$

$$(\text{resp. } \text{Gal}(L_R^+/k) = \bigoplus_{i=1}^q \bigoplus_{m_i} \text{Gal}(L_{R_i, m_i}^+/k)).$$

1.5. Representation of the bilinear general semigroup: Let L_v (resp. $L_{\bar{v}}$) be the set of pseudoramified real completions of L_L^+ (resp. L_R^+). Then, a bilinear general (or complete algebraic) semigroup over the product $L_{\bar{v}} \times L_v$ can be defined as the product of the (semi)group $T_n^t(L_{\bar{v}})$ of lower triangular matrices of order n over $L_{\bar{v}}$ by the (semi)group $T_n(L_v)$ of upper triangular matrices of order n over L_v according to [29]:

$$GL_n(L_{\bar{v}} \times L_v) = T_n^t(L_{\bar{v}}) \times T_n(L_v)$$

such that:

a) $GL_n(L_{\bar{v}} \times L_v)$ has the bilinear Gauss decomposition:

$$GL_n(L_{\bar{v}} \times L_v) = [(D_n(L_{\bar{v}}) \times D_n(L_v))[UT_n(L_v) \times UT_n^t(L_{\bar{v}})]$$

where

- $D_n(\cdot)$ is the subgroup of diagonal matrices;
- $UT_n(\cdot)$ is the subgroup of unitriangular matrices.

b) $GL_n(L_{\bar{v}} \times L_v)$ has for representation space $\text{Repsp}(GL_n(L_{\bar{v}} \times L_v))$ given by the tensor product $M_R \otimes M_L$ of a right $T_n^t(L_{\bar{v}})$ -semimodule M_R localized in the upper half space by a left $T_n(L_v)$ -semimodule M_L localized in the lower half space.

c) the left (resp. right) conjugacy classes of $T_n(L_v)$ (resp. $T_n^t(L_{\bar{v}})$) correspond to the left (resp. right) places v_i (resp. \bar{v}_i) of L_v (resp. $L_{\bar{v}}$).

Similarly, $GL_n(L_v^{nr} \times L_{\bar{v}}^{nr})$ has for representation space $\text{Repsp}(GL_n(L_v^{nr} \times L_{\bar{v}}^{nr}))$ given by the tensor product $M_R^{nr} \otimes M_L^{nr}$ of a right pseudounramified $T_n^t(L_{\bar{v}}^{nr})$ -semimodule M_R^{nr} by its left equivalent $T_n(L_v^{nr})$ -semimodule M_L^{nr} .

Considering complete bilinear algebraic (semi)groups is justified by the fact that they “cover” their “linear” equivalents. Indeed, it was proved in [29] that a linear complete

algebraic group $GL_n(L_{\bar{v}-v})$, with entries in $L_{\bar{v}-v} \equiv L_{\bar{v}} \cup L_v$ and representation space given by a vectorial space V of dimension n^2 , is covered by the bilinear complete algebraic semigroup $GL_n(L_{\bar{v}} \times L_v)$, having as representation space the $GL_n(L_{\bar{v}} \times L_v)$ -bisemimodule $M_R \otimes M_L$, at the conditions given in [29].

On the other hand, let $M_{L_{\oplus}}$ (resp. $M_{R_{\oplus}}$) denote the representation space of $T_n(L_{v_{\oplus}})$ (resp. $T_n^t(L_{\bar{v}_{\oplus}})$) with entries in the sum $L_{v_{\oplus}}$ (resp. $L_{\bar{v}_{\oplus}}$) of real pseudoramified completions $L_{v_{i,m_i}}$ (resp. $L_{\bar{v}_{i,m_i}}$).

Then, $M_{L_{\oplus}}$ (resp. $M_{R_{\oplus}}$) is homomorphic to M_L (resp. M_R) and decomposes into the direct sum of $T_n^t(L_{v_i})$ -subsemimodules M_{v_i} (resp. $T_n^t(L_{\bar{v}_i})$ -subsemimodules $M_{\bar{v}_i}$) according to:

$$M_{L_{\oplus}} = \bigoplus_{i=1}^q \bigoplus_{m_i} M_{v_i;m_i} \quad (\text{resp.} \quad M_{R_{\oplus}} = \bigoplus_{i=1}^q \bigoplus_{m_i} M_{\bar{v}_i;m_i})$$

such that:

- a) each $T_n(L_{v_i})$ -subsemimodule M_{v_i} (resp. $T_n^t(L_{\bar{v}_i})$ -subsemimodule $M_{\bar{v}_i}$) of dimension n constitutes a representative of the i -th conjugacy class of $T_n(L_v)$ (resp. $T_n^t(L_{\bar{v}})$).
- b) M_{v_i} and $M_{\bar{v}_i}$, $1 \leq i \leq q$, has a rank given by:

$$n_i = i^n \cdot N^n = f_{v_i}^n \cdot N^n.$$

Definition 1.6 Bisemimodules: The bilinear tensor product between the right $T_n^t(L_{\bar{v}})$ -semimodule M_R and the left $T_n(L_v)$ -semimodule M_L is given by [30]:

$$\begin{aligned} T_X : \quad \{M_R, M_L\} &\longrightarrow M_R \otimes M_L, \\ \{x_R, x_L\} &\longrightarrow x_R \times x_L, \quad \forall x_R \in M_R, x_L \in M_L, \end{aligned}$$

so that the pair $\{x_R, x_L\}$ of right and left points be mapped into the bipoint $x_R \times x_L$ characterized by a Riemanian signature [18]. $M_R \otimes M_L$ then is a $GL_n(L_{\bar{v}} \times L_v)$ -bisemimodule.

Similarly, the diagonal tensor product between the right and left semimodules M_R and M_L can be defined by

$$\begin{aligned} T_{X_D} : \quad \{M_R, M_L\} &\longrightarrow M_R \otimes_D M_L, \\ \{x_R, x_L\} &\longrightarrow x_R \times_D x_L, \end{aligned}$$

so that the “diagonal” bipoint $x_R \times_D x_L$ be characterized by a diagonal signature which can be Euclidian or not following that the metric be given by a diagonal unit matrix or by a diagonal matrix having diagonal elements taking values in the considered field.

$M_R \otimes_D M_L$ then is a $GL_n(L_{\bar{v}} \times_D L_v)$ -bisemimodule.

Definition 1.7 Bisemisheaves of rings: We want to introduce the set of smooth differentiable (bi)functions on the $GL_n(L_{\bar{v}} \times L_v)$ -bisemimodule $M_R \otimes M_L$, in such a way that these bifunctions are tensor products $\phi_{G_R}(x_{g_R}) \otimes \phi_{G_L}(x_{g_L})$ of smooth differentiable right functions $\phi_{G_R}(x_{g_R})$, $x_{g_R} \in T_n^t(L_{\bar{v}})$, on M_R , localized in the lower half space by symmetric smooth differentiable left functions $\phi_{G_L}(x_{g_L})$, $x_{g_L} \in T_n(L_v)$, on M_L , localized in the upper half space.

As $GL_n(L_{\bar{v}} \times L_v)$ is partitioned into conjugacy classes, we have to take into account the bifunctions $\phi_{G_{i,m_{i_R}}}(x_{i_R}) \otimes \phi_{G_{i,m_{i_L}}}(x_{i_L})$ on the conjugacy class representatives $M_{\bar{v}_{i,m_i}} \otimes M_{v_{i,m_i}}$. The set of smooth differentiable bifunctions $\{\phi_{G_{i,m_{i_R}}}(x_{i_R}) \otimes \phi_{G_{i,m_{i_L}}}(x_{i_L})\}_{i,m_i}$ on the $GL_n(L_{\bar{v}} \times L_v)$ -bisemimodule $M_R \otimes M_L$ is a bisemisheaf of rings noted $\widetilde{M}_R \otimes \widetilde{M}_L$ in such a way that this set of differentiable bifunctions are the (bi)sections of $\widetilde{M}_R \otimes \widetilde{M}_L$.

Indeed, \widetilde{M}_R (resp. \widetilde{M}_L), having as sections the smooth differentiable functions $\phi_{G_{i,m_{i_R}}}(x_{i_R})$ (resp. $\phi_{G_{i,m_{i_L}}}(x_{i_L})$), is a semisheaf of rings because it is a sheaf of abelian semigroups $\widetilde{M}_R(x_{i_R})$ (resp. $\widetilde{M}_L(x_{i_L})$) for every right (resp. left) point x_{i_R} (resp. x_{i_L}) of the topological semispace $M_R = \text{Repsp}(T_n^t(L_{\bar{v}}))$ (resp. $M_L = \text{Repsp}(T_n(L_v))$) where $\widetilde{M}_R(x_{i_R})$ (resp. $\widetilde{M}_L(x_{i_L})$) has the structure of a semiring.

The introduction of the bilinear Hilbert semispaces in the next section will concern the bisemisheaf of rings $\widetilde{M}_R \otimes_{(D)} \widetilde{M}_L$ as well as the $GL_n(L_{\bar{v}} \times L_v)$ -bisemimodule $M_R \otimes_{(D)} M_L$, on which it is defined, but the developments will only bear on $M_R \otimes_{(D)} M_L$ for the simplicity of the notations.

Definitions 1.8 a) External diagonal bilinear Hilbert semispaces \mathcal{H}_L^a and \mathcal{H}_R^a :

Let $M_R \otimes_D M_L$ be the diagonal $GL_n(L_{\bar{v}} \times_D L_v)$ -bisemimodule. Consider the projective linear mapping $p_L : M_R \otimes_D M_L \rightarrow M_{R(P)/L}$ projecting the $T_n^t(L_{\bar{v}})$ -semimodule M_R on the $T_n(L_v)$ -semimodule M_L . $M_{R(P)/L}$ is a bisemimodule representable locally by the bilinear Hilbert scheme $\text{Hilb}_{S_{R(P)}/S_L}$ (case $\widetilde{M}_{R(P)/L}$) [31].

If $M_{R(P)/L}$ is endowed with an external scalar product $\langle \phi_P, \psi \rangle$ defined from $M_{R(P)} \times_D M_L$ to \mathbb{C} , $\forall \phi_P \in M_{R(P)}$, $\forall \psi \in M_L$, this bisemimodule $M_{R(P)/L}$ will be called a left external bilinear Hilbert semispace, noted \mathcal{H}_L^a .

Similarly, if we consider the projective linear mapping $p_R : M_R \otimes_D M_L \rightarrow M_{L(P)/R}$ projecting the $T_n(L_v)$ -semimodule M_L on the $T_n^t(L_{\bar{v}})$ -semimodule M_R , we generate the bisemimodule $M_{L(P)/R}$ representable locally by the bilinear Hilbert scheme $\text{Hilb}_{S_{L(P)}/S_R}$.

Endowing $M_{L(P)/R}$ with an external scalar product from $M_{L(P)} \times_D M_R$ to \mathbb{C} , we shall get a right external bilinear Hilbert semispace noted \mathcal{H}_R^a .

Notice that \mathcal{H}_L^a and \mathcal{H}_R^a are characterized by ortho(normal) basis.

b) Internal diagonal bilinear Hilbert semispaces \mathcal{H}_a^- and \mathcal{H}_a^+ : Let $B_L : M_{R(P)} \rightarrow M_L$ (resp. $B_R : M_{L(P)} \rightarrow M_R$) be a bijective linear isometric map from $M_{R(P)}$ (resp. $M_{L(P)}$) to M_L (resp. M_R) mapping each covariant element of $M_{R(P)}$ (resp. $M_{L(P)}$) noted M_{L_R} (resp. M_{R_L}) into a contravariant element of M_L (resp. M_R).

Then, B_L (resp. B_R) transforms the left (resp. right) external Hilbert semispace \mathcal{H}_L^a (resp. \mathcal{H}_R^a) into the left (resp. right) internal bilinear Hilbert semispace \mathcal{H}_a^+ (resp. \mathcal{H}_a^-) in such a way that

- a) the bielements of \mathcal{H}_a^+ (resp. \mathcal{H}_a^-) are bivectors, i.e. two confounded vectors;
- b) each external scalar product of \mathcal{H}_L^a (resp. \mathcal{H}_R^a) is transformed into an internal scalar product defined from $M_{L_R} \times_D M_L$ (resp. $M_{R_L} \times_D M_R$) to \mathbb{C} .
- c) \mathcal{H}_a^+ and \mathcal{H}_a^- are characterized by ortho(normal) basis.

c) Extended external bilinear Hilbert semispaces H_L^a and H_R^a : If we consider on the non-Euclidian $GL_n(L_{\bar{v}} \times L_v)$ -bisemimodule $M_R \otimes M_L$ the projective linear mapping $p_L : M_R \otimes M_L \rightarrow M_{R(P)/cL}$ (“c” for complete), (resp. $p_R : M_R \otimes M_L \rightarrow M_{L(P)/cR}$) of the right (resp. left) semimodule M_R (resp. M_L) on the left (resp. right) semimodule M_L (resp. M_R), we get the non-Euclidian bisemimodule $M_{R(P)/cL}$ (resp. $M_{L(P)/cR}$).

If we endow $M_{R(P)/cL}$ (resp. $M_{L(P)/cR}$) with a complete external bilinear form defined from $M_{R(P)} \times M_L$ (resp. $M_{L(P)} \times M_R$) to \mathbb{C} , we get a left (resp. right) extended external bilinear Hilbert semispace noted H_L^a (resp. H_R^a) characterized by a non-Euclidian geometry and a non-orthogonal basis.

d) Extended internal bilinear Hilbert semispaces H_a^+ and H_a^- : The left (resp. right) extended external bilinear Hilbert semispace H_L^a (resp. H_R^a) can be transformed into the left (resp. right) extended internal bilinear Hilbert semispace H_a^+ (resp. H_a^-) by means of a bijective linear isometric map B_L (resp. B_R) from $M_{R(P)}$ (resp. $M_{L(P)}$) into M_L (resp. M_R).

The complete external bilinear form of H_L^a (resp. H_R^a) is then transformed into a complete internal bilinear form of H_a^+ (resp. H_a^-).

2 Cohomologies and representation spaces of algebras of operators

We are interested in the cohomology of compact spaces [8]. So, the most evident algebraic cohomology of compact spaces is the Eisenstein cohomology which is based upon the Borel-Serre compactification of the lattice space attached to an arithmetic group Γ . The Eisenstein cohomology classes were assumed to be represented by differential forms which are Eisenstein series [21], [22], [34].

Definition 2.1 The Shimura bisemivariety: Referring to the linear lattice space $X = GL_n(\mathbb{R})/GL_n(\mathbb{Z})$, [6], [7], a bilinear complex lattice bisemispace can be introduced by:

$$X_{S_{R \times L}} = GL_n(L_R^{(\mathbb{C})} \times L_L^{(\mathbb{C})})/GL_n((\mathbb{Z}/N\mathbb{Z})^2)$$

where

- $GL_n((\mathbb{Z}/N\mathbb{Z})^2)$ is a bilinear arithmetic semigroup over squares of integers modulo N ;
- $GL_n(L_R^{(\mathbb{C})} \times L_L^{(\mathbb{C})})$ is a bilinear algebraic semigroup with entries in the product $(L_R^{(\mathbb{C})} \times L_L^{(\mathbb{C})})$ of complex symmetric (semi)fields associated with $(L_R \times L_L)$.

The boundary $\partial \overline{X}_{S_{R \times L}}$ of the compactified bisemispace $\overline{X}_{S_{R \times L}}$ corresponds to the boundary of the Borel-Serre compactification and is given by:

$$\partial \overline{X}_{S_{R \times L}} = GL_n(L_{R_d}^+ \times L_{L_d}^+)/GL_n((\mathbb{Z}/N\mathbb{Z})^2) \approx GL_n(L_{\overline{v}} \times L_v)$$

where $L_{R_d}^+$ and $L_{L_d}^+$ are real compact semifields generated from L_R^+ and L_L^+ .

The double coset decomposition $\partial \overline{S}_{G_{R \times L}}$ of the boundary $\partial \overline{X}_{S_{R \times L}}$ of the compactified lattice bisemispace corresponds to a Shimura bisemivariety and is given by:

$$\partial \overline{S}_{G_{R \times L}} = P_n(L_{\overline{v}_1} \times L_{v_1}) \setminus GL_n(L_{R_d}^+ \times L_{L_d}^+)/GL_n((\mathbb{Z}/N\mathbb{Z})^2) \approx GL_n(L_{\overline{v}} \times L_v)$$

where

- the subgroup $GL_n((\mathbb{Z}/N\mathbb{Z})^2)$ constitutes the representation of the coset representatives of the tensor product $T_R(n; q) \otimes T_L(n; q)$ of Hecke operators [29];

- $P_n(L_{v_1})$ is the standard parabolic subsemigroup over the set $L_{v^1} = \{L_{v_1^1}, \dots, L_{v_{i,m_i}^1}, \dots, L_{v_{q,m_q}^1}\}$ of irreducible completions L_{v_{i,m_i}^1} having a rank N . $P_n(L_{\bar{v}^1} \times L_{v^1})$ is then a bilinear parabolic subsemigroup constituting the smallest connected pseudoramified normal bilinear subsemigroup of $GL_n(L_{\bar{v}} \times L_v)$ and representing the n -fold product $I_{L_{\bar{v}_i}} \times I_{L_{v_i}}$ of global inertia subgroups.

The double coset decomposition $\partial \bar{S}_{G_{R \times L}}$, corresponding to a Shimura bisemivariety and restricted to the lower (resp. upper) half space, becomes:

$$\begin{aligned} \partial \bar{S}_{G_R} &= P_n(L_{\bar{v}^1}) \setminus T_n^t(L_{R_d}^+) / T_n^t(\mathbb{Z}/N\mathbb{Z}) \\ (\text{resp. } \partial \bar{S}_{G_L} &= P_n(L_{v^1}) \setminus T_n(L_{L_d}^+) / T_n(\mathbb{Z}/N\mathbb{Z})). \end{aligned}$$

Proposition 2.2 *The (bi)cosets of the bilinear quotient semigroup $GL_n(L_{R_d}^+ \times L_{L_d}^+) / GL_n((\mathbb{Z}/N\mathbb{Z})^2)$ coincide with the conjugacy classes of the general bilinear semigroup $GL_n(L_{\bar{v}} \times L_v)$ with respect to the smallest connected pseudoramified normal bilinear subsemigroup given by the bilinear parabolic subsemigroup $P_n(L_{\bar{v}^1} \times L_{v^1})$.*

Sketch of the proof: According to 1.5, the conjugacy classes of $GL_n(L_{\bar{v}} \times L_v)$ are in one-to-one correspondence with the (bi)places of $L_{\bar{v}} \times L_v$. And, on the other hand, the bilinear subsemigroup $GL_n((\mathbb{Z}/N\mathbb{Z})^2)$ is a representation of the tensor product of Hecke operators such that the i -th (bi)coset representative of $GL_n((\mathbb{Z}/N\mathbb{Z})^2)$ corresponds to the biplace $\bar{v}_i \times v_i$ of $L_{\bar{v}} \times L_v$. ■

Proposition 2.3 *The bilinear cohomology (semi)group of the Shimura bisemivariety*

$$\partial \bar{S}_{G_{R \times L}} = P_n(L_{\bar{v}^1} \times L_{v^1}) \setminus GL_n(L_{R_d}^+ \times L_{L_d}^+) / GL_n((\mathbb{Z}/N\mathbb{Z})^2)$$

has its coefficient system given by the bisemisheaf $(\widetilde{M}_R^{2j} \otimes \widetilde{M}_L^{2j})$ and is given by the bilinear Eisenstein cohomology:

$$H^{2j}(\partial \bar{S}_{G_{R \times L}}, \widetilde{M}_R^{2j} \otimes \widetilde{M}_L^{2j}) \simeq \text{Repsp}(GL_{2j}(L_{\bar{v}} \times L_v)), \quad 2j \leq r,$$

which:

- *is in bijection with the representation space $\text{Repsp}(GL_{2j}(L_{\bar{v}} \times L_v))$ of the bilinear general semigroup $GL_{2j}(L_{\bar{v}} \times L_v)$;*
- *decomposes according to the conjugacy classes of $GL_{2j}(L_{\bar{v}} \times L_v)$.*

Proof:

1. It was demonstrated in [29] that the bilinear Eisenstein cohomology $H^n(\partial\overline{S}_{G_{R \times L}}, \widetilde{M}_R^{2j} \otimes \widetilde{M}_L^{2j})$ is in bijection with the representation of the bilinear general semigroup $GL_{2j}(L_{\overline{v}} \times L_v)$: this results from the fact that the Eisenstein bilinear cohomology can be deduced from the Weil bilinear algebra of the Lie bilinear nilpotent semialgebra.
2. As the bicosets of $\partial\overline{S}_{G_{R \times L}}^{(2j)} = P_{2j}(L_{\overline{v}^1} \times L_{v^1}) \setminus GL_{2j}(L_{R_d}^+ \times L_{L_d}^+) / GL_{2j}((\mathbb{Z}/N \mathbb{Z})^2)$ coincide with the conjugacy classes of $GL_{2j}(L_{\overline{v}} \times L_v)$, we have that the bilinear Eisenstein cohomology decomposes according to:

$$H^{2j}(\partial\overline{S}_{G_{R \times L}}, \widetilde{M}_{R_{\oplus}}^{2j} \otimes \widetilde{M}_{L_{\oplus}}^{2j}) \simeq \bigoplus_{i=1}^q \bigoplus_{m_i} (\widetilde{M}_{\overline{v}_i; m_i}^{2i} \otimes \widetilde{M}_{v_i; m_i}^{2i}) .$$

To each $T_{2j}(L_{v_i})$ -subsemimodule $M_{v_i}^{2j}$ (resp. $T_{2j}^t(L_{\overline{v}_i})$ -subsemimodule $M_{\overline{v}_i}^{2j}$) is associated a weight λ_{L_i} (resp. λ_{R_i}) characterizing the i -th left (resp. right) Hecke sublattice. Indeed, there exists the surjective morphism:

$$i_{M_{L,R}} : M_{L,R_{\oplus}}^{2j} \longrightarrow \Lambda_{L,R}$$

from the $T_{2j}(L_v)$ -semimodule M_L^{2j} (resp. $T_{2j}^t(L_{\overline{v}})$ -semimodule M_R^{2j}) into the $T_{2j}(\mathbb{Z}/N \mathbb{Z})$ -semimodule Λ_L (resp. $T_{2j}^t(\mathbb{Z}/N \mathbb{Z})$ -semimodule Λ_R) which is a left (resp. right) Hecke lattice decomposing according to the conjugacy classes of $T_{2j}(L_v)$ (resp. $T_{2j}^t(L_{\overline{v}})$):

$$\Lambda_L = \bigoplus_{i=1}^q \bigoplus_{m_i} \Lambda_{L_i; m_i} \quad (\text{resp.} \quad \Lambda_R = \bigoplus_{i=1}^q \bigoplus_{m_i} \Lambda_{R_i; m_i})$$

where $\Lambda_{L_i; m_i}$ (resp. $\Lambda_{R_i; m_i}$) is the i -th left (resp. right) Hecke sublattice having multiplicity $\text{sup}(m_i)$.

Indeed, on each left (resp. right) weight λ_{L_i} (resp. λ_{R_i}), which is a character of $\text{Rep}(T_{2j}(L_{v_i}))$ (resp. $\text{Rep}(T_{2j}^t(L_{\overline{v}_i}))$), there is the action of the Weyl semigroup W_L (resp. W_R) given by:

$$\phi(s_{i_L}) = w_{i_L} \lambda_{L_i} \quad (\text{resp.} \quad \phi(s_{i_R}) = w_{i_R} \lambda_{R_i})$$

where

- $\phi(s_{i_L})$ (resp. $\phi(s_{i_R})$) is a left (resp. right) Hecke character;
- $w_{i_L} \in W_L$, $w_{i_R} \in W_R$.

The left (resp. right) action of the Weyl group W_L (resp. W_R) consists in generating the multiplicities of the Hecke sublattices Λ_{L_i} (resp. Λ_{R_i}) to which correspond the subsemimodules M_{v_i, m_i} (resp. $M_{\bar{v}_i, m_i}$). ■

Corollary 2.4 *The general bilinear Eisenstein cohomology is characterized by the Künneth isomorphism:*

$$H_{R \times (D)L}^E : H^{2j}(\partial \bar{S}_{G_R}, \widetilde{M}_R^{2j}) \times_{(D)} H^{2j}(\partial \bar{S}_{G_L}, \widetilde{M}_L^{2j}) \xrightarrow{\sim} H^{2j}(\partial \bar{S}_{G_{R \times L}}, \widetilde{M}_R^{2j} \otimes_{(D)} \widetilde{M}_L^{2j}).$$

Sketch of proof: this is equivalent to defining the diagonal or complete product between a right and a left linear Eisenstein cohomology semigroup. ■

Definition 2.5 **Complete reducibility of $GL_{2n}(L_{\bar{v}} \times L_v)$ [29]:** Let

$$\begin{aligned} n_L &= 1_{1_L} + \cdots + 1_{k_L} + \cdots + 1_{\ell_L} + \cdots + 1_{n_L} \\ (\text{resp. } n_R &= 1_{1_R} + \cdots + 1_{k_R} + \cdots + 1_{\ell_R} + \cdots + 1_{n_R}) \end{aligned}$$

be a left (resp. right) partition of n_L (resp. n_R) labeling the irreducible representations of $T_{2n_L}(L_v)$ (resp. $T_{2n_R}(L_{\bar{v}})$).

Then,

$$1. \text{ Rep}(GL_{2n=2_1+\cdots+2_\ell+\cdots+2_n}(L_{\bar{v}} \times L_v)) = \bigsqcup_{2_\ell=2}^{2n} \text{Rep}(GL_{2_\ell}(L_{\bar{v}} \times L_v))$$

constitutes a completely reducible **orthogonal bilinear representation** of $GL_{2n}(L_{\bar{v}} \times L_v)$;

$$2. \text{ Rep}(GL_{2n_{R \times L}}(L_{\bar{v}} \times L_v))$$

$$= \bigsqcup_{2_{\ell_R}=2_{\ell_L}=2}^{2n} \text{Rep}(GL_{2_{\ell_R \times L}}(L_{\bar{v}} \times L_v)) \bigsqcup_{2_{k_R} \neq 2_{\ell_L}} \text{Rep}(T_{2_{k_R}}^t(L_{\bar{v}}) \times T_{2_{\ell_L}}(L_v)) ,$$

where $GL_{2_{\ell_R \times L}}$ is another notation for G_{2_ℓ} , constitutes a completely reducible **nonorthogonal bilinear representation** of $GL_{2n}(L_{\bar{v}} \times L_v)$.

Proposition 2.6 *Let \widetilde{M}_L^{2n} (resp. \widetilde{M}_R^{2n}) be a $2n$ -dimensional semisheaf on the $T_{2n}(L_v)$ -semimodule (resp. $T_{2n}^t(L_{\bar{v}})$ -semimodule).*

Let $\partial \bar{S}_{G_{2n=2_1+\cdots+2_n}}^{P_{2n=2_1+\cdots+2_n}}$ and $\partial \bar{S}_{G_{2n_R \times 2n_L}}^{P_{2n_R \times 2n_L}}$ denote respectively a completely reducible orthogonal and nonorthogonal Shimura bisemivariety instead of $\partial \bar{S}_{G_{R \times L}}$.

Then, the $2n$ -th bilinear Eisenstein cohomologies decompose into direct sums of completely irreducible orthogonal and nonorthogonal bilinear Eisenstein cohomologies according to:

$$\begin{aligned}
& \bullet H^{2n}(\partial \overline{S}_{G_{2n=2_1+\dots+2_n}}^{P_{2n=2_1+\dots+2_n}}, \widetilde{M}_R^{2n} \otimes_D \widetilde{M}_L^{2n}) \\
&= \bigoplus_{\ell_R=\ell_L} H^{2\ell}(\partial \overline{S}_{G_{2\ell_R, \ell_L}}^{P_{2\ell_R, \ell_L}}, \widetilde{M}_R^{2\ell_R} \otimes \widetilde{M}_L^{2\ell_L}) \\
&\simeq \text{Repsp}(GL_{2n=2_1+\dots+2_\ell+\dots+2_n}(L_{\overline{v}} \times L_v)) . \\
\\
& \bullet H^{2n}(\partial \overline{S}_{G_{2n_R \times 2n_L}}^{P_{2n_R \times 2n_L}}, \widetilde{M}_R^{2n} \otimes \widetilde{M}_L^{2n}) \\
&= \bigoplus_{\ell_R=\ell_L} H^{2\ell}(\partial \overline{S}_{G_{2\ell_R, 2\ell_L}}^{P_{2\ell_R, 2\ell_L}}, \widetilde{M}_R^{2\ell_R} \otimes \widetilde{M}_L^{2\ell_L}) \\
&\quad \bigoplus_{k_R \neq \ell_L} H^{2k_R, 2\ell_L}(\partial \overline{S}_{G_{2k_R \times 2\ell_L}}^{P_{2k_R \times 2\ell_L}}, \widetilde{M}_R^{2k_R} \otimes \widetilde{M}_L^{2\ell_L}) \\
&\simeq \text{Repsp}(GL_{2n_R \times L}(L_{\overline{v}} \times L_v))
\end{aligned}$$

where $\widetilde{M}_L^{2\ell_L}$ is a semisheaf on the $T_{2\ell_L}(L_v)$ -semimodule.

Proof:

1. The completely reducible orthogonal and nonorthogonal Shimura bisemivarieties are given respectively by:

$$\begin{aligned}
& \partial \overline{S}_{G_{2n=2_1+\dots+2_n}}^{P_{2n=2_1+\dots+2_n}} \\
&= \bigoplus_{\ell=1}^n \partial \overline{S}_{G_{2\ell_R, \ell_L}}^{P_{2\ell_R, \ell_L}} \\
&= \bigoplus_{\ell=1}^n P_{2\ell_R, \ell_L}(L_{\overline{v}^1} \times L_{v^1}) \setminus GL_{2\ell_R, \ell_L}(L_{R_d}^+ \times L_{L_d}^+) / GL_{2\ell_R, \ell_L}((\mathbb{Z}/N\mathbb{Z})^2)
\end{aligned}$$

and by:

$$\partial \overline{S}_{G_{2n_R \times 2n_L}}^{P_{2n_R \times 2n_L}} = \bigoplus_{\ell_R=\ell_L=1}^n \partial \overline{S}_{G_{2\ell_R, 2\ell_L}}^{P_{2\ell_R, 2\ell_L}} \bigoplus_{k_R \neq \ell_L=1}^n \partial \overline{S}_{G_{2k_R \times 2\ell_L}}^{P_{2k_R \times 2\ell_L}} .$$

2. The decomposition of the $2n$ -th bilinear Eisenstein cohomology into completely irreducible two-dimensional bilinear Eisenstein cohomologies results from its bijection with $\text{Rep}(GL_{2n=2_1+\dots+2_n}(L_{\overline{v}} \times L_v))$ or with $\text{Rep}(GL_{2n_R \times L}(L_{\overline{v}} \times L_v))$ according to definition 2.5.
3. Every two-dimensional Eisenstein bilinear cohomology decomposes with respect to the places in $(L_{\overline{v}} \times L_v)$ according to one-dimensional components:

$$H^{2\ell}(\partial \overline{S}_{G_{2\ell_R, 2\ell_L}}^{P_{2\ell_R, 2\ell_L}}, \widetilde{M}_R^{2\ell_R} \otimes \widetilde{M}_L^{2\ell_L}) \simeq \bigoplus_{i=1}^q \bigoplus_{m_i} (\widetilde{M}_{\overline{v}_i; m_i}^{2\ell_R} \otimes \widetilde{M}_{v_i; m_i}^{2\ell_L}) . \quad \blacksquare$$

Definition 2.7 Cuspidal representation in terms of global elliptic semimodules: The decomposition of the Eisenstein bilinear cohomology into one-dimensional irreducible components needs a cuspidal automorphic representation in terms of global elliptic bisemimodules.

Assume that f_L is a normalized eigenform (of a Hecke operator), holomorphic in the Poincare upper half plane H in \mathcal{C} , and defined in $\{\text{Im}(z_L) > 0\}$. f_L , expanded in formal power series $f_L = \sum_{i=1}^q a_{i_L} q_L^i$, where $q_L = e^{2\pi i z_L}$, $z_L \in \mathcal{C}$, is a cusp form of the space $S_L(N)$ and is an eigenvector of the Hecke operators T_{q_L} for $q \nmid N$ and U_{q_L} for $q \mid N$ where N is a positive integer. The Fourier coefficients a_{i_L} are eigenvalues $c(i, f_L)$ of Hecke operators such that $c(i, f_L)$ generate the ring of integers θ_L which leads to consider $S_L(N)$ as a θ_L -algebra.

The coalgebra $S_R(N)$ of $S_L(N)$, defined in the Poincare lower half plane H^* , is then composed of dual cusp forms $f_R = \sum_{i=1}^q a_{i_R} q_R^i$ with $q_R = e^{-2\pi i z_R}$, $z_R \in \mathcal{C}$, $z_R = z_L^*$, which are also eigenfunctions of Hecke operators T_{q_R} for $q \nmid N$ and U_{q_R} for $q \mid N$.

On the other hand, assume that the semisheaf $\widetilde{M}_{R,L}^{1_\ell}$ on the semimodule $M_{R,L}^{1_\ell}$ decomposes according to the conjugacy classes of $M_{R,L}^{1_\ell}$ into a set $s_{R,L} = \Gamma(\widetilde{M}_{R,L}^{1_\ell})$ of one-dimensional sections $s_{i_{R,L}}$, $1 \leq i \leq q \leq \infty$. For each section $s_{i_{R,L}} \in s_{R,L}$, let $\text{End}(G_{s_{R,L}})$ be the Frobenius endomorphism of the group $G_{s_{R,L}}$ of the $s_{i_{R,L}}$ and let $q_{R,L} \rightarrow q_{R,L}^i$ be the corresponding Frobenius substitution with $q_{R,L}^i = e^{\pm 2\pi i(i)x}$, $x \in \mathbb{R}$.

A global elliptic right (resp. left) $G_{s_{R,L}}$ -semimodule $\phi_{R,L}(s_{R,L})$ is a ring homomorphism: $\phi_{R,L} : s_{R,L} \rightarrow \text{End}(G_{s_{R,L}})$ defined by

$$\phi_{R,L}(s_{R,L}) = \sum_i \sum_{m_i} \phi(s_{i_{R,L}})_{i,m_i} q_{R,L}^i$$

where \sum_i runs over the one-dimensional sections of $\widetilde{M}_{R,L}^{1_\ell}$ and where \sum_{m_i} runs over the ideals of the decomposition group D_{i^2} of the biplace $\bar{v}_i \times v_i$.

Then, the space $S_{R,L}(\phi_{R,L})$ of global elliptic $G_{s_{R,L}}$ -semimodules $\phi_{R,L}(s_{R,L})$ is included into the space $S_{R,L}(N)$ of cusp forms $f_{R,L} : S_{R,L}(\phi_{R,L}) \hookrightarrow S_{R,L}(N)$ implying that $f_{R,L} \simeq \phi_{R,L}(s_{R,L})$.

Definition 2.8 The decomposition group: The ring of endomorphisms acting on the global elliptic $G_{s_{R,L}}$ -semimodules included into weight two cusp forms is generated over $\mathbb{Z}/N\mathbb{Z}$ by the Hecke operators $T_{q_{R,L}}$ for $q \nmid N$ and $U_{q_{R,L}}$ for $q \mid N$. The coset representatives of U_{q_L} are upper triangular and are given by the integral matrices $\begin{pmatrix} 1 & b_N \\ 0 & q_N \end{pmatrix}$ while the coset representatives of U_{q_R} are lower triangular and are given by the matrices $\begin{pmatrix} 1 & 0 \\ b_N & q_N \end{pmatrix}$. For a general integer $r = a \cdot d$, we would have respectively the integral matrices

$\begin{pmatrix} a_N & b_N \\ 0 & d_N \end{pmatrix}$ and $\begin{pmatrix} a_N & 0 \\ b_N & d_N \end{pmatrix}$ of determinant $r \cdot N^2 \simeq a_N \cdot d_N$ (class “0” of integers modulo N). These integral matrices modulo N are defined by considering that $q_N = * \bmod N \simeq q \cdot N$ and $b_N = * \bmod N$. Taking into account that the unipotent radical $u(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and its transposed $u(b)^t = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ generate \mathbb{F}_q , the following coset representative

$$GL_2((\mathbb{Z}/N\mathbb{Z})_{|q^2}^2) = \left[\begin{pmatrix} 1 & b_N \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_N & 1 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ 0 & q_N^2 \end{pmatrix}$$

will be adopted for the tensor product $U_{q_R} \otimes U_{q_L}$ of Hecke operators where $\alpha_{q_N^2} = \begin{pmatrix} 1 & 0 \\ 0 & q_N^2 \end{pmatrix}$ is the split Cartan subgroup and where $D_{q_N^2, b_N} = \begin{pmatrix} 1 & b_N \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_N & 1 \end{pmatrix}$ is the representation of the decomposition group associated with $\alpha_{q_N^2}$. Then, $GL_2((\mathbb{Z}/N\mathbb{Z})_{|q^2}^2)$ corresponds to a Gauss decomposition of the class “ q_N^2 ”.

Proposition 2.9 *The eigenvalues $\lambda_{\pm}(q_N^2, b_N^2)$ of the coset representatives $GL_2(\mathbb{Z}/N\mathbb{Z})_{|q^2}^2$ of $U_{q_R} \times U_{q_L}$ are such that they are coefficients of the global elliptic $G_{s_{R,L}}$ -semimodules $\phi_{R,L}(s_{R,L})$, i.e. $\phi(s_{q_{R,L}})_{q,b} \equiv \lambda_{\pm}(q_N^2, b_N^2)$. Then, the one-dimensional components of the global elliptic semimodule $\phi_{R,L}(s_{R,L})$ are one-dimensional semitori localized respectively in the upper and in the lower half space and characterized by radii given by $r(q_N^2, b_N^2) = (\lambda_+(q_N^2, b_N^2) - \lambda_-(q_N^2, b_N^2))/2$.*

Proof: The eigenvalues of $GL_2(\mathbb{Z}/N\mathbb{Z})_{|q^2}^2 = \alpha_{q_N^2} \cdot D_{q_N^2, b_N^2}$ are

$$\lambda_{\pm}(q_N^2, b_N^2) = \frac{(1 + b_N^2 + q_N^2) \pm [(1 + b_N^2 + q_N^2)^2 - 4q_N^2]^{\frac{1}{2}}}{2}$$

and verify

$$\begin{aligned} \text{Trace}(GL_2(\mathbb{Z}/N\mathbb{Z})_{|q^2}^2) &= 1 + b_N^2 + q_N^2, \\ \text{and } \det(GL_2(\mathbb{Z}/N\mathbb{Z})_{|q^2}^2) &= \lambda_+(q_N^2, b_N^2) \cdot \lambda_-(q_N^2, b_N^2) = q_N^2. \end{aligned} \quad \blacksquare$$

Proposition 2.10 *According to the Langlands bilinear global program [29] and proposition 2.6, every two-dimensional Eisenstein bilinear cohomology is in bijection with a global elliptic $G_{s_R \times s_L}$ -bisemimodule $\phi_R(s_R) \otimes \phi_L(s_L)$:*

$$\begin{aligned} H^{2\ell}(\partial \overline{S}_{G_{2\ell_R, \ell_L}}^{P_{2\ell_R, \ell_L}}, \widetilde{M}_{R_{\oplus}}^{2\ell_R} \otimes \widetilde{M}_{L_{\oplus}}^{2\ell_L}) &\simeq \bigoplus_{q=1}^q \bigoplus_{m_i} \left(\widetilde{M}_{v_i, m_i}^{1\ell_R} \otimes \widetilde{M}_{v_i, m_i}^{1\ell_L} \right) \approx \phi_R(s_R) \otimes \phi_L(s_L) \\ &= \sum_{i=1}^q \sum_{m_i} \lambda_+(i_N^2, m_i^2) e^{-2\pi i(i)x} \otimes \sum_{i=1}^q \sum_{m_i} \lambda_-(i_N^2, m_i^2) e^{2\pi i(i)x} \end{aligned}$$

in such a way that the i -th bisection on the $GL_{2\ell}(L_{\bar{v}_i} \times L_{v_i})$ -subbisemimodule $(M_R^{2\ell_R} \otimes M_L^{2\ell_L})$ in $H^{2\ell}(\partial \bar{S}_{G^{2\ell_R, 2\ell_L}}, \widetilde{M}_R^{2\ell_R} \otimes \widetilde{M}_L^{2\ell_L})$ be in one-to-one correspondence with the set of m_i biterms $\{\lambda_+(i_N^2, m_i^2)e^{-2\pi i(i)x} \times \lambda_-(i_N^2, m_i^2)e^{2\pi i(i)x}\}_{m_i}$ of the global elliptic bisemimodule $\phi_R(s_R) \otimes \phi_L(s_L)$. So, the global elliptic bisemimodule constitutes the cuspidal representation of the Eisenstein bilinear cohomology.

Definition 2.11 The semialgebra of elliptic operators $\text{Ell}_{R,L}(\widetilde{M}_{R,L})$ is the semialgebra of linear differential operators $D_{R,L}$ defined on the space $\Gamma_{R,L}(\widetilde{M}_{R,L})$ of smooth sections $s_{R,L}$ of $\widetilde{M}_{R,L}$ and having their principal symbol $\sigma(D_{R,L})$ invertible [3].

Proposition 2.12 The bilinear Hilbert semispace H_a^\mp is the natural representation space for the semialgebra of elliptic operators.

Proof: Taking into account the $B_L \circ p_L$ (resp. $B_R \circ p_R$) map as introduced in definitions 1.8, the bisemisheaf $\widetilde{M}_R \otimes_{(D)} \widetilde{M}_L$ on the $GL_n(L_{\bar{v}} \times L_v)$ -bisemimodule $M_R \otimes_{(D)} M_L$ will be transformed into an extended internal or internal left (resp. right) Hilbert bisemisheaf according to:

$$\begin{aligned} B_L \circ p_L &: \quad \widetilde{M}_R \otimes_{(D)} \widetilde{M}_L \longrightarrow \widetilde{M}_{L_R} \otimes_{(D)} \widetilde{M}_L \equiv \widetilde{\mathcal{M}}_L, \\ B_R \circ p_R &: \quad \widetilde{M}_R \otimes_{(D)} \widetilde{M}_L \longrightarrow \widetilde{M}_R \otimes_{(D)} \widetilde{M}_{L_R} \equiv \widetilde{\mathcal{M}}_R. \end{aligned}$$

Consequently, $\text{Ell}_R(\widetilde{M}_R) \otimes_{(D)} \text{Ell}_L(\widetilde{M}_L)$ acting on $\widetilde{M}_{L_R} \otimes_{(D)} \widetilde{M}_L$ (resp. $\widetilde{M}_R \otimes_{(D)} \widetilde{M}_{L_R}$) will be an algebra of bioperators (or a bisemialgebra of operators) acting on an extended internal or internal left (resp. right) bilinear Hilbert semispace H_a^+ (resp. H_a^-) or \mathcal{H}_a^+ (resp. \mathcal{H}_a^-) and will be noted:

$$\begin{aligned} &[\text{Ell}_R(\widetilde{M}_R) \otimes \text{Ell}_L(\widetilde{M}_L)](H_a^\pm) \\ \text{or} \quad &[\text{Ell}_R(\widetilde{M}_R) \otimes_D \text{Ell}_L(\widetilde{M}_L)](\mathcal{H}_a^\pm). \end{aligned}$$

On the other hand, a semialgebra of operators $\text{Ell}_{R,L}(\widetilde{M}_{R,L})$ acting on H_a^\mp or \mathcal{H}_a^\mp will be given by $\text{Ell}_{R,L}(\widetilde{M}_{R,L})(H_a^\mp)$ or $\text{Ell}_{R,L}(\widetilde{M}_{R,L})(\mathcal{H}_a^\mp)$ in such a way that $\text{Ell}_R(\widetilde{M}_R)$ (resp. $\text{Ell}_L(\widetilde{M}_L)$) be a semialgebra of right (resp. left) elliptic linear operators acting on the set of sections of the semisheaf \widetilde{M}_R (resp. \widetilde{M}_L) over the $G_R(L_{\bar{v}})$ (resp. $G_L(L_v)$)-semimodule M_R (resp. M_L) of H_a^\mp or \mathcal{H}_a^\mp , where G_R (resp. G_L) is another notation for T_n^t (resp. T_n).

Taking into account the considerations given about the enveloping algebras in definition 1.1, it then becomes clear that the extended bilinear Hilbert semispace H_a^\pm is the natural representation space for the bisemialgebra and the semialgebra of elliptic operators. ■

Definitions 2.13 **a) Semialgebra of bounded operators:** If $\mathcal{L}_{R,L}^B(\widetilde{M}_{R,L})$ denotes the semialgebra of right (resp. left) operators on the semisheaf $\widetilde{M}_{R,L}$ over the $G_{R,L}(L_{\overline{v},v})$ -semimodule $M_{R,L}$, then the semialgebra of right (resp. left) self-adjoint bounded operators $T_{R,L}$ on H_a^\mp or \mathcal{H}_a^\mp will be given by: $\mathcal{L}_{R,L}^B(H_a^\mp)$ and $\mathcal{L}_{R,L}^B(\mathcal{H}_a^\mp)$, while the bisemialgebra of self-adjoint bounded operators on H_a^\mp and on \mathcal{H}_a^\mp will be: $(\mathcal{L}_R^B \otimes \mathcal{L}_L^B)(H_a^\mp)$ and $(\mathcal{L}_R^B \otimes_D \mathcal{L}_L^B)(\mathcal{H}_a^\mp)$ such that the right and left self-adjoint bounded operators $T_{R,L} \in \mathcal{L}_{R,L}^B$ act respectively on the right and left semisheaves of H_a^\mp and \mathcal{H}_a^\mp .

b) A weight on a semialgebra $\mathcal{L}_{R,L}^B(H_a^+)$ is given by the positive bilinear form $(T_{R s_{i_{L_R}}}, s_{i_L})$ or $(s_{i_{L_R}}, T_L s_{i_L})$ which is a map from $\mathcal{L}_{R,L}^B(\widetilde{M}_{L_R} \times \widetilde{M}_L)$ into \mathcal{C} for every section $s_{i_{L_R}} \in \widetilde{M}_{L_R}$ and $s_{i_L} \in \widetilde{M}_L$.

Similarly, a weight on a bisemialgebra $(\mathcal{L}_R^B \otimes \mathcal{L}_L^B)(H_a^+)$ will be given by the positive bilinear form $(T_{R s_{i_{L_R}}}, T_L s_{i_L})$ which is a map from $(\mathcal{L}_R^B(\widetilde{M}_{L_R}) \times \mathcal{L}_L^B(\widetilde{M}_L))$ into \mathcal{C} for all $T_{R,L} \in \mathcal{L}_{R,L}^B$.

Definition 2.14 **Complex analytic semivariety:** Let \overline{X}_{S_R} (resp. \overline{X}_{S_L}) denote the right (resp. left) complex semispace compactified from $X_{S_R} = GL_n(L_R^{\mathcal{G}})/GL_n(\mathbb{Z}/N\mathbb{Z})$ (resp. $X_{S_L} = GL_n(L_L^{\mathcal{G}})/GL_n(\mathbb{Z}/N\mathbb{Z})$) being the right (resp. left) complex (lattice) analytic semivariety introduced in section 2.1 and covered by $\partial\overline{X}_{S_R}$ (resp. $\partial\overline{X}_{S_L}$) or by $\partial\overline{S}_{G_R}$ (resp. $\partial\overline{S}_{G_L}$).

Let $\widetilde{M}_{R,L}^s$ be an analytic semisheaf on \overline{X}_{S_R} (resp. \overline{X}_{S_L}).

Then, the analytic de Rham cohomology $H^*(\overline{X}_{S_{R,L}}, \widetilde{M}_{R,L}^s)$ can be computed through the analytic de Rham complex taking into account that:

Lemma 2.15 *There is an isomorphism between the (algebraic) Eisenstein cohomology $H^*(\partial\overline{S}_{G_{R,L}}, \widetilde{M}_{R,L})$ and the analytic de Rham cohomology $H^*(\partial\overline{X}_{S_{R,L}}, \widetilde{M}_{R,L}^s)$.*

Proof: Indeed, the isomorphism between the following two de Rham cohomologies of Ω^* -smooth differential forms with respect to $\partial\overline{S}_{G_{R,L}}$ and $\overline{X}_{S_{R,L}}$ [20], [8], [12]:

$$H^*(\Omega_{\partial\overline{S}_{G_{R,L}}}^*) \simeq H^*(\Omega_{\overline{X}_{S_{R,L}}}^*)$$

leads naturally to the following isomorphism:

$$H^*(\partial\overline{S}_{G_{R,L}}, \widetilde{M}_{R,L}) \simeq H^*(\overline{X}_{S_{R,L}}, \widetilde{M}_{R,L}^s). \quad \blacksquare$$

Definition 2.16 **Analytic bilinear Hilbert semispaces:** From the complete (resp. diagonal) bilinear tensor [30] product between the right and left analytic semisheaves \widetilde{M}_R^s

and \widetilde{M}_L^s , we can construct a left (resp. right) analytic bisemisheaf $\widetilde{\mathcal{M}}_{L(D)}^s$ (resp. $\widetilde{\mathcal{M}}_{R(D)}^s$) of a left (resp. right) analytic bilinear Hilbert semispace H_h^+ or \mathcal{H}_h^+ (resp. H_h^- or \mathcal{H}_h^-) in complete analogy with which was done in definition 1.8.

Proposition 2.17 *The analytic bilinear Hilbert semispace H_h^\pm is the natural representation space for the (bi)semialgebras of elliptic operators: $\text{Ell}_{R,L}(\widetilde{M}_{R,L}^s)$ and $(\text{Ell}_R(\widetilde{M}_R^s) \otimes_{(D)} \text{Ell}_L(\widetilde{M}_L^s))$.*

Proof: This results from definitions 2.12 and 1.1. ■

Definitions 2.18 a) **Serre-Swan theorem:** Let $\widetilde{M}_{R,L}^{\text{top}} = C(X_{R,L})$ be the semi-algebra of continuous functions on a compact (semi)space $X_{R,L}$. We shall denote by $\text{VEC}(X_{R,L})$ the category of complex vector bundles over $X_{R,L}$ and $P(\widetilde{M}_{R,L}^{\text{top}})$ the category of finitely generated projective right (resp. left) semimodules $P_{R,L}^{\text{top}}$ over $\Gamma(\text{VEC}(X_{R,L}))$.

Then, the Serre-Swan theorem asserts that the categories $\text{VEC}(X_{R,L})$ and $P_{R,L}^{\text{top}}(\Gamma(\text{VEC}(X_{R,L})))$ are equivalent [36].

b) **The bisemialgebra $C(X_R \times_{(D)} X_L)$:** Let $x_{R,L}$ be a right (resp. left) point of the right (resp. left) semialgebra $\widetilde{M}_{R,L}^{\text{top}}$.

The complete (resp. diagonal) tensor product between the right and left semialgebras $\widetilde{M}_R^{\text{top}}$ and $\widetilde{M}_L^{\text{top}}$ can be defined by:

$$\begin{aligned} T_X^{\text{top}} : \quad \{ \widetilde{M}_R^{\text{top}}, \widetilde{M}_L^{\text{top}} \} &\longrightarrow \widetilde{M}_R^{\text{top}} \otimes_{(D)} \widetilde{M}_L^{\text{top}}, \\ \{ x_R, x_L \} &\longrightarrow x_R \times_{(D)} x_L, \end{aligned}$$

so that the bipoint $x_R \times_{(D)} x_L$ be characterized by a complete (resp. diagonal) signature.

$\widetilde{M}_R^{\text{top}} \otimes_D \widetilde{M}_L^{\text{top}}$ is then a finitely generated bisemialgebra.

c) **Topological bilinear Hilbert semispace:** By application of the $(B_L \circ p_L)$ (resp. $B_R \circ p_R$) linear map, the bisemialgebra $(\widetilde{M}_R^{\text{top}} \otimes_{(D)} \widetilde{M}_L^{\text{top}})$ can be transformed into an extended internal or internal left (resp. right) topological Hilbert bisemisheaf $\widetilde{\mathcal{M}}_{L(D)}^{\text{top}}$ (resp. $\widetilde{\mathcal{M}}_{R(D)}^{\text{top}}$) which becomes an extended internal or internal left (resp. right) topological bilinear Hilbert semispace H_{top}^\mp or $\mathcal{H}_{\text{top}}^\mp$ if it is endowed with a complete or a diagonal bilinear form with values in \mathcal{C} .

Proposition 2.19 *The extended internal and internal left (resp. right) topological bilinear Hilbert semispaces H_{top}^\pm and $\mathcal{H}_{\text{top}}^\pm$ are C^* -(bi)semialgebras.*

Proof: By definition $\widetilde{M}_{R,L}^{\text{top}}$ is a right (resp. left) semialgebra $C(X_{R,L})$ of continuous sections $s_{R,L}^{\text{top}}(X_{R,L})$ on $X_{R,L}$.

Now, the bisemialgebra $\widetilde{\mathcal{M}}_{L(D)}^{\text{top}}$ or $\widetilde{\mathcal{M}}_{R(D)}^{\text{top}}$ is an involutive bisemialgebra over \mathcal{C} of continuous bifunctions $s_R^{\text{top}}(X_R) \otimes_{(D)} s_L^{\text{top}}(X_L)$. Indeed, the involution, which must be taken into account, is a bilinear map transforming H_{top}^+ or $\mathcal{H}_{\text{top}}^+$ (resp. H_{top}^- or $\mathcal{H}_{\text{top}}^-$) into H_{top}^- or $\mathcal{H}_{\text{top}}^-$ (resp. H_{top}^+ or $\mathcal{H}_{\text{top}}^+$).

Recall the composition of maps:

$$\begin{aligned} B_L \circ p_L &: \quad \widetilde{M}_R^{\text{top}} \otimes_{(D)} \widetilde{M}_L^{\text{top}} \longrightarrow \widetilde{M}_{LR}^{\text{top}} \otimes_{(D)} \widetilde{M}_L^{\text{top}} \equiv \widetilde{\mathcal{M}}_{L(D)}^{\text{top}}, \\ B_R \circ p_R &: \quad \widetilde{M}_R^{\text{top}} \otimes_{(D)} \widetilde{M}_L^{\text{top}} \longrightarrow \widetilde{M}_{RL}^{\text{top}} \otimes_{(D)} \widetilde{M}_R^{\text{top}} \equiv \widetilde{\mathcal{M}}_{R(D)}^{\text{top}}, \end{aligned}$$

as introduced in definitions 1.8.

So, the bilinear map:

$$\begin{aligned} (p_L^{-1} \circ B_L^{-1}) \otimes_{(D)} (B_R \circ p_R) &: \quad \widetilde{\mathcal{M}}_{L(D)}^{\text{top}} \longrightarrow \widetilde{\mathcal{M}}_{R(D)}^{\text{top}}, \\ (B_R \circ p_R) \otimes_{(D)} (p_L^{-1} \circ B_L^{-1}) &: \quad \widetilde{\mathcal{M}}_{R(D)}^{\text{top}} \longrightarrow \widetilde{\mathcal{M}}_{L(D)}^{\text{top}}, \end{aligned}$$

transforms the bisemialgebra $\widetilde{\mathcal{M}}_{L(D)}^{\text{top}}$ (resp. $\widetilde{\mathcal{M}}_{R(D)}^{\text{top}}$) into the bisemialgebra $\widetilde{\mathcal{M}}_{R(D)}^{\text{top}}$ (resp. $\widetilde{\mathcal{M}}_{L(D)}^{\text{top}}$) which corresponds to an antilinear involution transforming the left (resp. right) bilinear Hilbert semispace H_{top}^+ or $\mathcal{H}_{\text{top}}^+$ (resp. H_{top}^- or $\mathcal{H}_{\text{top}}^-$) into the right (resp. left) involuted bilinear Hilbert semispace H_{top}^- or $\mathcal{H}_{\text{top}}^-$ (resp. H_{top}^+ or $\mathcal{H}_{\text{top}}^+$). ■

Definitions 2.20 a) K -functor of Kasparov [25]: We are now interested in extensions of the bisemialgebra $\widetilde{M}_{R,L}^{\text{top}}$. Let $\mathcal{L}_{R,L}^B(\widetilde{M}_{R,L}^{\text{top}})$ denote the semialgebra of bounded operators on $\widetilde{M}_{R,L}^{\text{top}}$ and let $\mathcal{K}_{R,L}$ be the ideal of compact operators.

The set of extension classes of $\mathcal{K}_{R,L}$ by $\mathcal{L}_{R,L}^B(\widetilde{M}_{R,L}^{\text{top}})$, noted $\text{Ext}(\mathcal{L}_{R,L}^B(\widetilde{M}_{R,L}^{\text{top}}), \mathcal{K}_{R,L})$, is an abelian semigroup naturally isomorphic to $\text{Ext}(X_{R,L})$ as developed by Brown, Douglas and Fillmore [10], [11].

In connection with the work of Atiyah [4], [5], G.G. Kasparov constructed a general K -functor $K_*K^*(\widetilde{M}_{R,L}^{\text{top}}, \mathcal{L}_{R,L}^B)$, special cases of which are the ordinary cohomological K -functor $K^*(\widetilde{M}_{R,L}^{\text{top}})$ and the homological K -functor $K_*(\mathcal{L}_{R,L}^B)$.

Especially interesting is the case where the C^* -semialgebras $\widetilde{M}_{R,L}^{\text{top}}$ and $\mathcal{L}_{R,L}^B$ are equipped with the continuous action of a locally compact semigroup $G_{R,L}^\ell$. This allows to define an abelian group $KK^{G_{R,L}^\ell}(\widetilde{M}_{R,L}^{\text{top}}, \mathcal{L}_{R,L}^B)$ [26].

b) Bisemialgebra of bounded operators: Considering the C^* -bisemialgebra $\widetilde{\mathcal{M}}_{L(D)}^{\text{top}}$ (resp. $\widetilde{\mathcal{M}}_{R(D)}^{\text{top}}$), the bisemialgebra of bounded operators on it will be $(\mathcal{L}_R^B \otimes_{(D)} \mathcal{L}_L^B)(\widetilde{\mathcal{M}}_{L(D)}^{\text{top}})$

(resp. $(\mathcal{L}_R^B \otimes_{(D)} \mathcal{L}_L^B)(\widetilde{\mathcal{M}}_{R(D)}^{\text{top}})$) or $(\mathcal{L}_R^B \otimes_{(D)} \mathcal{L}_L^B)(H_{\text{top}}^\mp)$ (resp. $(\mathcal{L}_R^B \otimes_{(D)} \mathcal{L}_L^B)(\mathcal{H}_{\text{top}}^\mp)$) if we envisage their actions on the extended (resp. diagonal) bilinear Hilbert semispace H_{top}^\mp (resp. $\mathcal{H}_{\text{top}}^\mp$).

3 Von Neumann semialgebras and bisemialgebras

Definitions 3.1 a) Norm topology of bounded operators: Let $(\mathcal{L}_R^B \otimes \mathcal{L}_L^B)(H_{\text{top}}^+)$ be the bisemialgebra of bounded operators acting from the topological extended bilinear Hilbert semispace H_{top}^+ into itself.

Then, the norm topology for an operator $T_R \otimes T_L \in \mathcal{L}_R^B \otimes \mathcal{L}_L^B$ will be defined by

$$\|T_R \otimes T_L\| = \sup (\|T_R s_{L_R}^{\text{top}} \times T_L s_L^{\text{top}}\| / \|s_{L_R}^{\text{top}} \times s_L^{\text{top}}\|) ,$$

for every section $s_{L_R}^{\text{top}} \in \widetilde{M}_{L_R}^{\text{top}}$ and $s_L^{\text{top}} \in \widetilde{M}_L^{\text{top}} \subset H_{\text{top}}^+$, since, if $\mathcal{L}_L^B(H_{\text{top}}^+)$ is the semialgebra of left bounded operators acting on the semisheaf $\widetilde{M}_L^{\text{top}}$ of H_{top}^+ , the norm topology for a left bounded operator T_L is given by

$$\|T_L\| = \sup (\|T_L s_L^{\text{top}}\| / \|s_L^{\text{top}}\|) .$$

b) An involution on the operator $T_{R,L}$ is defined by

$$i_L : T_R \longrightarrow T_R^\dagger \equiv T_L ,$$

$$i_R : T_L \longrightarrow T_L^\dagger \equiv T_R ,$$

such that $(T_R^\dagger s_{L_R}^{\text{top}}, T_L^\dagger s_L^{\text{top}}) = (T_R s_{L_R}^{\text{top}}, T_L s_L^{\text{top}})$ making T_R and T_L self-adjoint.

Definitions 3.2 Bisemialgebras of von Neumann on extended bilinear Hilbert semispaces: **a)** A right (resp. left) semialgebra of von Neumann $\mathbb{M}_{R,L}(H_{\text{top}}^\mp)$ in the topological extended bilinear Hilbert semispace H_{top}^\mp is an involutive subalgebra of $\mathcal{L}_{R,L}^B(H_{\text{top}}^\mp)$ having a closed norm topology [19].

Similarly, a semialgebra of von Neumann $\mathbb{M}_{R,L}(H_h^\mp)$ in H_h^\mp is an involutive subsemialgebra of $\text{Ell}_{R,L}(\widetilde{M}_{R,L}^s)$ having a closed norm topology.

b) A bisemialgebra of von Neumann $\mathbb{M}_{R \times L}(H_{\{h\}}^\mp)$ in $H_{\{h\}}^\mp$ is an involutive subbisemialgebra of $(\mathcal{L}_R^B \otimes \mathcal{L}_L^B)(H_{\{h\}}^\mp)$ having a closed norm topology [19].

c) A bisemialgebra of von Neumann $\mathbb{M}_{R \times L}(H_a^\mp)$ in the algebraic extended bilinear Hilbert semispace H_a^\mp is also an involutive subbisemialgebra of $(\mathcal{L}_R^B \otimes \mathcal{L}_L^B)(H_a^\mp)$ having a closed norm topology.

Proposition 3.3 *Let $\mathbb{M}_{R,L}(H_a^\mp)$ and $\mathbb{M}_{R \times L}(H_a^\mp)$ be respectively a semialgebra and a bisemialgebra of von Neumann on the algebraic bilinear extended Hilbert semispace H_a^\mp .*

Let $\mathbb{M}_{R,L}(H_h^\mp)$ and $\mathbb{M}_{R \times L}(H_h^\mp)$ be respectively a semialgebra and a bisemialgebra of von Neumann on the analytic bilinear extended Hilbert semispace H_h^\mp .

Then, we have the isomorphisms:

$$\begin{aligned} i_{\mathbb{M}_{R,L}^a - \mathbb{M}_{R,L}^h} &: \mathbb{M}_{R,L}(H_a^\mp) \longrightarrow \mathbb{M}_{R,L}(H_h^\mp), \\ i_{\mathbb{M}_{R \times L}^a - \mathbb{M}_{R \times L}^h} &: \mathbb{M}_{R \times L}(H_a^\mp) \longrightarrow \mathbb{M}_{R \times L}(H_h^\mp). \end{aligned}$$

Proof: This results immediately from lemma 2.15. ■

Proposition 3.4 *There exists an isomorphism*

$$i_{M_{\{h\}^{\text{top}}} - \mathbb{M}_{R,L}^{\{h\}^{\text{top}}}} : M_{\{h\}^{\text{top}}} \longrightarrow \mathbb{M}_{R,L}^{\{h\}^{\text{top}}} (H_{\{h\}^{\text{top}}}^\mp)$$

between an algebra of von Neumann $M_{\{h\}^{\text{top}}}$ on a linear Hilbert space $h_{\{h\}^{\text{top}}}$ [26] and a semialgebra of von Neumann $\mathbb{M}_{R,L}^{\{h\}^{\text{top}}} (H_{\{h\}^{\text{top}}}^\mp)$ on the extended bilinear Hilbert semispace $H_{\{h\}^{\text{top}}}^\mp$.

Proof: Let $V_{\{h\}^{\text{top}}}$ be a compact manifold of class C^0 (resp. C^∞) associated with $M_R \otimes_{(D)} M_L$ in the sense of [29] and let $A_{\{h\}^{\text{top}}}$ be the corresponding stellar algebra of C^0 (resp. C^∞) functions on $V_{\{h\}^{\text{top}}}$ with values in \mathcal{C} .

Then, a Fredholm module on $A_{\{h\}^{\text{top}}}$ is essentially given by the involutive representation $\Pi_{\{h\}^{\text{top}}}$ of $A_{\{h\}^{\text{top}}}$ in a linear Hilbert space $h_{\{h\}^{\text{top}}}$ and by a self-adjoint operator F .

Furthermore, an algebra of von Neumann $M_{\{h\}^{\text{top}}}$ in a linear Hilbert space $h_{\{h\}^{\text{top}}}$ is an involutive subalgebra of bounded operators $\mathcal{L}(h_{\{h\}^{\text{top}}})$ from $h_{\{h\}^{\text{top}}}$ to $h_{\{h\}^{\text{top}}}$ such that $M_{\{h\}^{\text{top}}}$ be $\sigma(\mathcal{L}(h_{\{h\}^{\text{top}}}), \mathcal{L}(h_{\{h\}^{\text{top}}})_*)$ closed.

Now, it is clear that there is a one-to-one correspondence between:

- a) a Fredholm module on $A_{\{h\}^{\text{top}}}$ and a subsemialgebra of $\mathcal{L}_{R,L}^B(H_{\{h\}^{\text{top}}}^\mp)$ since the extended bilinear Hilbert semispace $H_{\{h\}^{\text{top}}}^\mp$ can be considered as a representation space of the linear Hilbert space $h_{\{h\}^{\text{top}}}$ covered by $H_{\{h\}^{\text{top}}}^\mp$ [29].
- b) the weak topological condition of closeness of $\sigma(\mathcal{L}(h_{\{h\}^{\text{top}}}), \mathcal{L}(h_{\{h\}^{\text{top}}})_*)$ and the condition of closed norm topology of $\mathcal{L}_{R,L}^B(H_{\{h\}^{\text{top}}}^\mp)$ since $\mathcal{L}(h_{\{h\}^{\text{top}}})_*$ is the dual of $\mathcal{L}(h_{\{h\}^{\text{top}}})$.

As there is an isomorphism between a Fredholm module on $A_{\{h}^{\text{top}}$ and a subspace of $\mathcal{L}(h_{\{h}^{\text{top}})$, we have the announced isomorphism $i_{M_{\{h}^{\text{top}} - \mathbb{M}_{R,L}^{\{h}^{\text{top}} : M_{\{h}^{\text{top}} \rightarrow \mathbb{M}_{R,L}^{\{h}^{\text{top}} (H_{\{h}^{\text{top}})} . \blacksquare$

3.5. Shifted actions of differential bioperators on the representation spaces of bilinear semigroups:

1. Let $T_{R,L}^{D_m} \in \mathbb{M}_{R,L}(H_a^\mp)$ be a right (resp. left) differential linear operator of rank m (i.e. operating on m variables) of the semialgebra of von Neumann $\mathbb{M}_{R,L}(H_a^\pm)$. This operator $T_R^{D_m}$ (resp. $T_L^{D_m}$), noted in condensed form T_R^D (resp T_L^D), is assumed to be associated with the action of a $T_m^t(\mathbb{R})$ -semigroup structure (resp. a $T_m(\mathbb{R})$ -semigroup structure) on the right (resp. left) n -dimensional semisheaf \widetilde{M}_R (resp. \widetilde{M}_L) of the extended bilinear Hilbert semispace H_a^+ , with $m \leq n$. Recall [3] that a $T_m^t(\mathbb{R})$ -semigroup structure (resp. a $T_m(\mathbb{R})$ -semigroup structure) on \widetilde{M}_R (resp. \widetilde{M}_L) means a principal $T_m^t(\mathbb{R})$ -bundle (resp. a $T_m(\mathbb{R})$ -bundle) on \widetilde{M}_R (resp. \widetilde{M}_L).
2. Similarly, $(T_R^D \otimes T_L^D)$ will denote the tensor product of the right and left differential operators (T_R^D and T_L^D) acting on the bisemisheaf $(\widetilde{M}_R \otimes \widetilde{M}_L)$ such that $(T_R^D \otimes T_L^D) \in \mathbb{M}_{R \times L}(H_a^\pm)$ be associated with a principal $GL_m(\mathbb{R} \times \mathbb{R}) = T_m(\mathbb{R}) \times T_m^t(\mathbb{R})$ -bundle on $(\widetilde{M}_R \otimes \widetilde{M}_L)$.
3. Let $(T_R^D \times T_L^D)$ be the tensor product of a right and a left linear differential operator of rank m such that the action of $(T_R^D \otimes T_L^D)$ be associated with a $GL_m(\mathbb{R} \times \mathbb{R})$ -principal bundle on the bisemisheaf $(\widetilde{M}_R \otimes \widetilde{M}_L)$ over the $GL_n(L_{\overline{v}} \times L_v)$ -bisemimodule $(M_R \otimes M_L)$. Then, the action of $(T_R^D \otimes T_L^D)$ on $(\widetilde{M}_R \otimes \widetilde{M}_L)$ is equivalent to:

(a) consider the mapping

$$T_R^D \otimes T_L^D : \quad \widetilde{M}_R \otimes \widetilde{M}_L \longrightarrow \widetilde{M}_{R_{n[m]}} \otimes \widetilde{M}_{L_{n[m]}}$$

from the bisemisheaf $(\widetilde{M}_R \otimes \widetilde{M}_L)$ over the $GL_n(L_{\overline{v}} \times L_v)$ -bisemimodule $(M_R \otimes M_L)$ to the bisemisheaf $(\widetilde{M}_{R_{n[m]}} \otimes \widetilde{M}_{L_{n[m]}})$ over the $GL_{n[m]}((L_{\overline{v}} \otimes \mathbb{R}) \times (L_v \otimes \mathbb{R}))$ -bisemimodule $(M_{R_{n[m]}} \otimes M_{L_{n[m]}})$ such that $(\widetilde{M}_{R_{n[m]}} \otimes \widetilde{M}_{L_{n[m]}})$ be a bisemisheaf shifted into $(m \times m)$ dimensions.

(b) consider a shift into $(m \times m)$ dimensions of the functional representation space $\text{FRepsp}(GL_n(L_{\overline{v}} \times L_v))$ of the general bilinear semigroup $GL_n(L_{\overline{v}} \times L_v)$ leading to the homomorphism:

$$T_R^D \otimes T_L^D : \quad \text{FRepsp}(GL_n(L_{\overline{v}} \times L_v)) \longrightarrow \text{FRepsp}(GL_{n[m]}((L_{\overline{v}} \otimes \mathbb{R}) \times (L_v \otimes \mathbb{R})))$$

where $\text{FRepsp}(GL_{n[m]}((L_{\bar{v}} \otimes \mathbb{R}) \times (L_v \otimes \mathbb{R})))$, denoting the functional representation space of $GL_n(L_{\bar{v}} \times L_v)$ shifted into $(m \times m)$ dimensions, is the shifted bisemisheaf $(\widetilde{M}_{R_{n[m]}} \otimes \widetilde{M}_{L_{n[m]}})$ on the bisemigroupoid $GL_{n[m]}((L_{\bar{v}} \otimes \mathbb{R}) \times (L_v \otimes \mathbb{R}))$ and is equal to:

$$\begin{aligned} \text{FRepsp}(GL_{n[m]}((L_{\bar{v}} \otimes \mathbb{R}) \times (L_v \otimes \mathbb{R}))) \\ = \text{AdFRepsp}(GL_m(\mathbb{R} \times \mathbb{R})) \times \text{FRepsp}(GL_n(L_{\bar{v}} \otimes L_v)) \end{aligned}$$

in such a way that [32]

- $\text{AdFRepsp}(GL_m(\mathbb{R} \times \mathbb{R}))$, being the adjoint functional representation space of $GL_m(\mathbb{R} \times \mathbb{R})$, corresponds to the action of $(T_R^D \times T_L^D)$;
- $\text{FRepsp}(GL_n(L_{\bar{v}} \otimes L_v))$, being the functional representation space of $GL_n(L_{\bar{v}} \times L_v)$, correspond to the bisemisheaf $(\widetilde{M}_R \otimes \widetilde{M}_L)$.

4. Similarly, the shifting “action” of $(T_R^D \otimes T_L^D)$ on functional representation space of the bilinear subsemigroup $GL_n((\mathbb{Z}/N\mathbb{Z})^2)$ would be:

$$\begin{aligned} T_R^D \otimes T_L^D : \\ \text{FRepsp}(GL_n((\mathbb{Z}/N\mathbb{Z})^2)) \\ = \text{FRepsp}(D_n((\mathbb{Z}/N\mathbb{Z})^2) \times [UT_n^t(\mathbb{Z}/N\mathbb{Z}) \times UT_n(\mathbb{Z}/N\mathbb{Z})]) \\ \longrightarrow \text{FRepsp}(GL_{n[m]}((\mathbb{Z}/N\mathbb{Z})^2 \otimes \mathbb{R}^2)) \\ = \text{FRepsp}(D_{n[m]}((\mathbb{Z}/N\mathbb{Z})^2 \otimes \mathbb{R}^2) \\ \times [UT_{n[m]}^t((\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{R}^2) \times UT_{n[m]}((\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{R}^2)]) \end{aligned}$$

where:

- $\text{FRepsp}(D_{n[m]}((\mathbb{Z}/N\mathbb{Z})^2 \otimes \mathbb{R}^2))$ is the functional representation space of the subgroup of integer diagonal matrices of order n shifted into m dimensions.
- $\text{FRepsp}(UT_{n[m]}((\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{R}))$ is the functional representation space of the subgroup of integer unitriangular matrices shifted in m dimensions.

5. And, the functional representation space of bilinear parabolic subsemigroup $P_n(L_{\bar{v}^1} \times L_{v^1})$ would also be shifted into $(m \times m)$ dimensions under the action of $(T_R^D \otimes T_L^D)$

according to:

$$\begin{aligned}
& T_R^D \otimes T_L^D : \\
& \text{FRepsp}(P_n((L_{\bar{v}^1} \times L_{v^1})) = \text{FRepsp}(D_n(L_{\bar{v}^1} \times L_{v^1}) \times [UT_n^t(L_{\bar{v}^1}) \times UT_n(L_{v^1})]) \\
& \longrightarrow \text{FRepsp}(P_{n[m]}((L_{\bar{v}^1} \otimes \mathbb{R}) \times (L_{v^1} \otimes \mathbb{R}))) \\
& = \text{FRepsp}(D_{n[m]}((L_{\bar{v}^1} \otimes \mathbb{R}) \times (L_{v^1} \otimes \mathbb{R})) \\
& \quad \times [UT_{n[m]}^t(L_{\bar{v}^1} \otimes \mathbb{R}) \times UT_{n[m]}(L_{v^1} \otimes \mathbb{R})]) .
\end{aligned}$$

6. On the other hand, referring to section 1.5, $GL_n(L_{\bar{v}}^{nr} \times L_v^{nr})$ has $GL(M_R^{nr} \otimes M_L^{nr}) \equiv \Gamma_R \times \Gamma_L$ for bilinear (semi)group of automorphisms and has for pseudounramified conjugacy classes the biclasses $\gamma(i)_R \times \gamma(i)_L$, $1 \leq i \leq q$, if the set of fixed bielements is the smallest normal bilinear subsemigroup $P_n(L_{\bar{v}^1}^{nr} \times L_{v^1}^{nr})$ of $M_R^{nr} \otimes M_L^{nr}$. This smallest normal bilinear subsemigroup of $GL_n(L_{\bar{v}}^{nr} \times L_v^{nr})$ is the n -dimensional equivalent of the product, right by left, of the global inertia subgroups $I_{L_{\bar{v}_i}} \times I_{L_{v_i}}$ of degree $N^2 = 1$ as introduced in definition 1.4 [27].

In this context, the action of $(T_R^D \otimes T_L^D)$ on $(\widetilde{M}_R^{nr} \otimes \widetilde{M}_L^{nr})$, associated with the principal $GL_m(\mathbb{R} \times \mathbb{R})$ -bundle on $(\widetilde{M}_R^{nr} \otimes \widetilde{M}_L^{nr})$ with group $GL_m(\mathbb{R} \times \mathbb{R})$, leads to envisage that the bilinear semigroupoid $GL_{n[m]}((L_{\bar{v}}^{nr} \otimes \mathbb{R}) \times (L_v^{nr} \otimes \mathbb{R}))$, shifting in $(m \times m)$ dimensions, has $GL(\widetilde{M}_{R[n]}^{nr} \otimes \widetilde{M}_{L[n]}^{nr}) \equiv \Gamma_R^{[m]} \times \Gamma_L^{[m]}$ for bilinear semigroupoid of shifted automorphisms and has for pseudounramified conjugacy classes the biclasses $(\gamma^{[m]}(i)_R \times \gamma^{[m]}(i)_L)$ shifted in $(m \times m)$ dimensions, if the set of shifted fixed bielements corresponds to the smallest normal bilinear subsemigroupoid $P_{n[m]}((L_{\bar{v}^1}^{nr} \otimes \mathbb{R}) \times (L_{v^1}^{nr} \otimes \mathbb{R}))$, i.e. the bilinear pseudounramified parabolic subsemigroupoid.

The shifted pseudounramified conjugacy biclasses $(\gamma^{[m]}(i)_R \times \gamma^{[m]}(i)_L)$ are in one-to-one correspondence with their unshifted equivalents $(\gamma(i)_R \times \gamma(i)_L)$ because the bilinear subsemigroup $(\Gamma_R^{[m]} \times \Gamma_L^{[m]})$ of automorphisms shifting in $(m \times m)$ real dimensions results from the principal $GL_m(\mathbb{R} \times \mathbb{R})$ -bundle on $(\widetilde{M}_R^{nr} \times \widetilde{M}_L^{nr})$ and corresponds to the $(m \times n)$ -dimensional representation of the product, right by left, of the differential Galois semigroups of the algebraic extensions $L_R^{nr,+}$ and $L_L^{nr,+}$.

7. $GL_n(L_{\bar{v}} \times L_v)$ has for bilinear subsemigroup of automorphisms $\text{Pra } \Gamma_R \times \text{Pra } \Gamma_L$ and has for pseudoramified conjugacy classes the biclasses $g(i)_R \times g(i)_L$ if the set of fixed bielements is of dimension $N > 1$ with respect to the basis of $M_R \otimes M_L$. These fixed bielements of $g(i)_R \times g(i)_L$ correspond to the product, right by left, of completions of degrees equal to $N > 1$.

Similarly, $GL_{n[m]}((L_{\bar{v}} \otimes \mathbb{R}) \times (L_v \otimes \mathbb{R}))$ has for bilinear subsemigroup of shifted

automorphisms $(\text{Pra } \Gamma_R^{[m]} \times \text{Pra } \Gamma_L^{[m]})$ and has for shifted pseudoramified conjugacy classes the biclasses $(g^{[m]}(i)_R \times g^{[m]}(i)_L)$.

As $(\text{Pra } \Gamma_R^{[m]} \times \text{Pra } \Gamma_L^{[m]})$ is the bilinear subsemigroupoid $GL(\widetilde{M}_{R_{n[m]}} \otimes \widetilde{M}_{L_{n[m]}})$ of automorphisms shifting in $(m \times m)$ real dimensions with respect to the biaction of $(T_R^D \otimes T_L^D)$ on $(\widetilde{M}_R \otimes \widetilde{M}_L)$, associated with the $GL_m(\mathbb{R} \times \mathbb{R})$ -principal bundle introduced in 3.), it is clear that the shifted pseudoramified conjugacy biclasses $(g^{[m]}(i)_R \times g^{[m]}(i)_L)$ are in one-to-one correspondence with the unshifted pseudoramified conjugacy biclasses $g(i)_R \times g(i)_L$.

Proposition 3.6 *The action of the differential bioperator $(T_R^D \otimes T_L^D)$ of rank $(m \times m)$, associated with a principal $GL_m(\mathbb{R} \times \mathbb{R})$ -bundle on the $(n \times n)$ -dimensional pseudo(un)-ramified bisemisheaf $(\widetilde{M}_R^{(nr)} \otimes \widetilde{M}_L^{(nr)})$, consists in mapping $(\widetilde{M}_{R_{n[m] \oplus}}^{(nr)} \otimes \widetilde{M}_{L_{n[m] \oplus}}^{(nr)})$ into $(\widetilde{M}_{R_{n[m] \oplus}}^{(nr)} \otimes \widetilde{M}_{L_{n[m] \oplus}}^{(nr)})$ shifted into $(m \times m)$ dimensions:*

$$T_R^D \otimes T_L^D : \quad \widetilde{M}_{R_{n[m] \oplus}}^{(nr)} \otimes \widetilde{M}_{L_{n[m] \oplus}}^{(nr)} \longrightarrow (\widetilde{M}_{R_{n[m] \oplus}}^{(nr)} \otimes \widetilde{M}_{L_{n[m] \oplus}}^{(nr)})$$

such that:

- a) $\widetilde{M}_{R_{n[m] \oplus}} \otimes \widetilde{M}_{L_{n[m] \oplus}}$ decomposes into shifted pseudoramified subbisemisheaves according to the shifted pseudoramified conjugacy biclasses $g_R^{[m]}(i) \times g_L^{[m]}(i)$ of the bisemigroupoid $GL_{n[m]}((L_{\overline{v}} \otimes \mathbb{R}) \times (L_v \otimes \mathbb{R}))$ and with respect to the shifted automorphisms $\text{Pra } \Gamma_R^{[m]} \times \text{Pra } \Gamma_L^{[m]}$ of $GL_{n[m]}((L_{\overline{v}} \otimes \mathbb{R}) \times (L_v \otimes \mathbb{R}))$ as follows:

$$\widetilde{M}_{R_{n[m] \oplus}} \otimes \widetilde{M}_{L_{n[m] \oplus}} = \bigoplus_{i=1}^q \bigoplus_{m_i} (\widetilde{M}_{R_{n[m]}}(i) \otimes \widetilde{M}_{L_{n[m]}}(i))$$

where the integer q is related to the dimension $(q \cdot N)^n$ of the algebraic basis of $\widetilde{M}_{R_{n[m]}}(q)$ and $\widetilde{M}_{L_{n[m]}}(q)$, i.e. to the number of Galois automorphisms.

- b) $\widetilde{M}_{R_{n[m] \oplus}}^{nr} \otimes \widetilde{M}_{L_{n[m] \oplus}}^{nr}$ decomposes into shifted pseudounramified subbisemisheaves according to the shifted pseudounramified conjugacy biclasses $\gamma_R^{[m]}(i) \times \gamma_L^{[m]}(i)$ of $GL_{n[m]}((L_{\overline{v}}^{nr} \otimes \mathbb{R}) \times (L_v^{nr} \otimes \mathbb{R}))$ as follows:

$$\widetilde{M}_{R_{n[m] \oplus}}^{nr} \otimes \widetilde{M}_{L_{n[m] \oplus}}^{nr} = \bigoplus_{i=1}^q \bigoplus_{m_i} (\widetilde{M}_{R_{n[m]}}^{nr}(i) \otimes \widetilde{M}_{L_{n[m]}}^{nr}(i))$$

where the integer q , i.e. the global class residue degree $f_{v_q} = q$ (see definition 1.3), refers to the algebraic dimension q^n of $\widetilde{M}_{R_{n[m]}}^{nr}(q)$ and of $\widetilde{M}_{L_{n[m]}}^{nr}(q)$.

Proof:

1. The shifted bisemisheaf $(\widetilde{M}_{R_{n[m]}}^{(nr)} \otimes \widetilde{M}_{L_{n[m]}}^{(nr)})$ is a biobject of the derived category $D(\widetilde{M}_R \otimes \widetilde{M}_L, \mathbb{R} \otimes \mathbb{R})$.
2. The algebraic dimension $(q \cdot N)^n$ of $M_{R_{n[m]}}(q)$ and of $M_{L_{n[m]}}(q)$ corresponds to the number of Galois automorphisms while the algebraic dimension $(q \cdot N)^m$ corresponds to the number of shifted automorphisms.
3. The pseudounramified algebraic dimension q^n is such that q corresponds to the number of archimedean places of the semifields L_L^+ and L_R^+ . ■

Definition 3.7 Pseudoramified and pseudounramified algebraic dimensions:

Until now, two kinds of algebraic dimensions have emerged:

- a) the “pseudoramified” algebraic dimensions $i^n \cdot N^n$, referring to the Galois extension degrees being multiples of $N > 1$.

The shifted pseudoramified algebraic dimensions $i^m \cdot N^m$ referring to the dimensions of the m -dimensional representations of the differential Galois subgroups;

- b) the pseudounramified algebraic dimension i^n referring to the n -th powers of the global residue degree i .

The pseudounramified algebraic dimension i^m referring to the dimensions of the m -dimensional representations of the corresponding differential Galois subgroups.

Consider for example the left $T_n(L_{v_i})$ -subsemimodule $M_{v_i} \subset M_L$ (see section 1.5) having a rank $n_i = i^n \cdot N^n = f_{v_i}^n \cdot N^n$. Then, the **pseudoramified algebraic dimension** of M_{v_i} is equal to its rank $n_i = i^n \cdot N^n$.

Note that the geometric dimension of the $T_n(L_{v_i})$ -subsemimodule M_{v_i} is equal to “ n ”. So, the geometric and algebraic dimensions generally do not coincide.

Proposition 3.8 *Let $\widetilde{M}_R^{(nr)} \otimes \widetilde{M}_L^{(nr)}$ denote the pseudo(un)ramified bisemisheaf over the real $GL_n(L_{\overline{v}}^{(nr)} \times L_v^{(nr)})$ -bisemimodule $(M_R^{(nr)} \otimes M_L^{(nr)})$ isomorphic to its analytic counterpart $(M_R^{s(nr)} \otimes M_L^{s(nr)})$.*

Let $(T_R^D \otimes T_L^D)$ be a differential bioperator acting on $(\widetilde{M}_R^{(nr)} \otimes \widetilde{M}_L^{(nr)})$ and transforming them into the corresponding shifted bisemisheaves $(\widetilde{M}_{R_{n[m]}}^{(nr)} \otimes \widetilde{M}_{L_{n[m]}}^{(nr)})$.

Then, the bisemimodules $(M_R^{(nr)} \otimes M_L^{(nr)})$ as well as their shifted counterparts $(M_{R_{n[m]}}^{(nr)} \otimes M_{L_{n[m]}}^{(nr)})$ are characterized by the following ranks or algebraic dimensions:

- a) the pseudounramified bisemimodule $M_R^{nr} \otimes M_L^{nr}$ has for algebraic dimension $d = \sum_{i=1}^d i^{n^2}$;
- b) pseudounramified shifted bisemimodule $M_{R_{n[m]}}^{nr} \otimes M_{L_{n[m]}}^{nr}$ has for algebraic dimension $d = \sum_{i=1}^d i^{n^2}$ and for shifted algebraic dimension $d_s = \sum_{i=1}^q i^{m^2}$;
- c) pseudoramified bisemimodule $M_R \otimes M_L$ has for algebraic dimension $d = \sum_{i=1}^d (i \cdot N)^{n^2}$;
- d) pseudoramified shifted bisemimodule $M_{R_{[m]}} \otimes M_{L_{n[m]}}$ has for algebraic dimension $d = \sum_{i=1}^d (i \cdot N)^{n^2}$ and for shifted algebraic dimension $d_s = \sum_{i=1}^q (i \cdot N)^{m^2}$.

Proof: This results from sections 3.5 and 3.6 and from [30]. ■

Proposition 3.9 Under the “action” of the bioperator $(T_R^D \otimes T_L^D) \in \mathbb{M}_{R \times L}(H_a^\pm)$ of rank $(m \times m)$, the Shimura bisemivariety

$$\partial \overline{S}_{G_{R \times L}} = P_n(L_{\overline{v}^1}^+ \times L_{v^1}^+) \setminus GL_n(L_{R_d}^+ \times L_{L_d}^+) / GL_n((\mathbb{Z} / N \mathbb{Z})^2)$$

is shifted into $(m \times m)$ dimensions according to:

$$T_R^D \otimes T_L^D : \partial \overline{S}_{G_{R \times L}} \longrightarrow \partial \overline{S}_{G_{R \times L; n[m]}}$$

where $\partial \overline{S}_{G_{R \times L; n[m]}}$ is the shifted Shimura bisemivariety given by:

$$\begin{aligned} \partial \overline{S}_{G_{R \times L; n[m]}} &= P_{n[m]}((L_{\overline{v}^1} \otimes \mathbb{R}) \times (L_{v^1} \otimes \mathbb{R})) \setminus \\ &GL_{n[m]}((L_{R_d}^+ \otimes \mathbb{R}) \times (L_{L_d}^+ \otimes \mathbb{R})) / GL_{n[m]}((\mathbb{Z} / N \mathbb{Z})^2 \otimes \mathbb{R}^2) . \end{aligned}$$

Proposition 3.10 The bilinear cohomology semigroup of the Shimura bisemivariety $\partial \overline{S}_{G_{R \times L}}$ is shifted under the action of the differential bioperator $(T_R^D \otimes T_L^D) \in \mathbb{M}_{R \times L}(H_a^\pm)$ according to:

$$\begin{aligned} T_R^D \otimes T_L^D : H^{2j}(\partial \overline{S}_{R \times L}, \widetilde{M}_R^{2j} \otimes \widetilde{M}_L^{2j}) \\ \longrightarrow H^{2j-2k}(\partial \overline{S}_{G_{R \times L; n[m]}}, \widetilde{M}_{R_{2j[2k]}}^{2j} \otimes \widetilde{M}_{L_{2j[2k]}}^{2j}) \end{aligned}$$

in such a way that the shifted bilinear Eisenstein cohomology decomposes according to the bicosets of the quotient bisemigroupoid

$$GL_{2j[2k]}((L_{\overline{v}} \otimes \mathbb{R}) \times (L_v \otimes \mathbb{R})) / GL_{2j[2k]}((\mathbb{Z} / N \mathbb{Z})^2 \otimes \mathbb{R}^2)$$

as follows:

$$H^{2j-2k}(\partial\overline{S}_{G_{R \times L; n[m]}}, \widetilde{M}_{R_{2j[2k] \oplus}}^{2j} \otimes \widetilde{M}_{L_{2j[2k] \oplus}}^{2j}) = \bigoplus_{i=1}^q \bigoplus_{m_i} (\widetilde{M}_{R_{2j[2k]}}^{2j}(i; m_i) \otimes \widetilde{M}_{L_{2j[2k]}}^{2j}(i; m_i))$$

where m_i refers to the multiplicity of the shifted subbisemimodule $(\widetilde{M}_{R_{2j[2k]}}^{2j}(i; m_i) \otimes \widetilde{M}_{L_{2j[2k]}}^{2j}(i; m_i))$.

Proof: According to proposition 2.3 and the Langlands bilinear global program developed in [29] and in [32], we have that

$$\begin{aligned} H^{2j}(\partial\overline{S}_{G_{R \times L}}, \widetilde{M}_{R_{\oplus}}^{2j} \otimes \widetilde{M}_{L_{\oplus}}^{2j}) &\approx \text{FRepsp}(GL_{2j}(L_{\overline{v}_{\oplus}} \times L_{v_{\oplus}})) \\ &= \bigoplus_{i=1}^q \bigoplus_{m_i} (\widetilde{M}_{\overline{v}_i; m_i}^{2j} \otimes \widetilde{M}_{v_i; m_i}^{2j}). \end{aligned}$$

Then, the shifted bilinear Eisenstein cohomology verifies:

$$\begin{aligned} H^{2j-2k}(\partial\overline{S}_{G_{R \times L; n[m]}}, \widetilde{M}_{R_{2j[2k] \oplus}}^{2j} \otimes \widetilde{M}_{L_{2j[2k] \oplus}}^{2j}) &\simeq \text{FRepsp}(GL_{2j[2k]}((L_{\overline{v}_{\oplus}} \otimes \mathbb{R}) \times (L_{v_{\oplus}} \otimes \mathbb{R}))) \\ &= \bigoplus_{i=1}^q \bigoplus_{m_i} (\widetilde{M}_{R_{2j[2k]}}^{2j}(i; m_i) \otimes (\widetilde{M}_{L_{2j[2k]}}^{2j}(i; m_i)) \end{aligned}$$

such that:

$$\text{FRepsp}(GL_{2j[2k]}((L_{\overline{v}_i} \otimes \mathbb{R}) \times (L_{v_i} \otimes \mathbb{R}))) = \widetilde{M}_{R_{2j[2k]}}^{2j}(i; m_i) \otimes \widetilde{M}_{L_{2j[2k]}}^{2j}(i; m_i). \quad \blacksquare$$

Proposition 3.11 *Let us fix the integers*

$$1 \leq \ell_{R,L} \leq j, \quad 1 \leq k_R \leq j \quad \text{and} \quad 1 \leq u_{R,L} \leq k, \quad 1 \leq v_R \leq k$$

with the condition that $m \leq n$.

Then, the shifted bilinear Eisenstein cohomology decomposes into the direct sum of completely irreducible orthogonal or nonorthogonal shifted bilinear Eisenstein cohomologies according to:

- $H^{2j-2k}(\partial\overline{S}_{G_{R \times L; n[m]}}, \widetilde{M}_{R_{2j[2k] \oplus}}^{2j} \otimes_D \widetilde{M}_{L_{2j[2k] \oplus}}^{2j})$
 $= \bigoplus_{i=1}^q \bigoplus_{m_i} \bigoplus_{\ell_R = \ell_L} \bigoplus_{u_R, L} H^{2\ell_R - 2u_R}(\partial\overline{S}_{G_{R \times L; n[m]}}, \widetilde{M}_{R_{2\ell_R[2u_R]}}^{2j}(i; m_i) \otimes \widetilde{M}_{L_{2\ell_L[2u_L]}}^{2j}(i; m_i))$
 $= \text{FRepsp}(GL_{2j=2_1+\dots+2_\ell+\dots+2_{j[k]}}((L_{\overline{v}} \otimes \mathbb{R}) \times (L_v \otimes \mathbb{R})))$
- $H^{2j-2k}(\partial\overline{S}_{G_{R \times L; n[m]}}, \widetilde{M}_{R_{2j[2k] \oplus}}^{2j} \otimes \widetilde{M}_{L_{2j[2k] \oplus}}^{2j})$

$$\begin{aligned}
&= \bigoplus_{i=1}^q \bigoplus_{m_i} \bigoplus_{\ell_R=\ell_L} \bigoplus_{u_R, L} H^{2\ell_R-2u_R}(\partial \overline{S}_{G_{R \times L}; n[m]}, \widetilde{M}_{R_{2\ell_R}[2u_R]}^2(i; m_i) \otimes \widetilde{M}_{L_{2\ell_L}[2u_L]}^2(i; m_i)) \\
&\quad \bigoplus_{i=1}^q \bigoplus_{m_i} \bigoplus_{k_R \neq \ell_L} \bigoplus_{v_R \neq u_L} H^{2k_R-2v_R}(\partial \overline{S}_{G_{R \times L}; n[m]}, \widetilde{M}_{R_{2k_R}[2v_R]}^2(i; m_i) \otimes \widetilde{M}_{L_{2\ell_L}[2u_L]}^2(i; m_i)) \\
&= \text{FRepsp}(GL_{2j_{R \times L}[2k]}((L_{\overline{v}} \otimes \mathbb{R}) \times (L_v \otimes \mathbb{R}))
\end{aligned}$$

where $\widetilde{M}_{2\ell_L[2u_L]}^2(i; m_i)$ is a two-dimensional shifted functional representation space over the $T_{2\ell_L[u_L]}(L_{v_i} \otimes \mathbb{R})$ -semimodule.

Proof: This proposition introduces the complete reducibility of the bilinear Eisenstein shifted cohomology semigroup in complete analogy with the unshifted case developed in proposition 2.6 and accordig to [32]. \blacksquare

Definition 3.12 Solvable bilinear Hilbert semispaces:

1. Let $\widetilde{M}_R^{(nr)} \otimes \widetilde{M}_L^{(nr)} = \{\widetilde{M}_{\overline{v}_i, m_i}^{(nr)} \otimes \widetilde{M}_{v_i, m_i}^{(nr)}\}_{i=1}^q$ be the bisemisheaf of differentiable bifunc-tions

$$\widetilde{M}_{\overline{v}_i, m_i}^{(nr)} \otimes \widetilde{M}_{v_i, m_i}^{(nr)} \equiv \phi_{G_i, m_i_R}(x_{i_R}) \otimes \phi_{G_i, m_i_L}(x_{i_L})$$

over the $GL_n(L_{\overline{v}}^{(nr)} \times L_v^{(nr)})$ -bisemimodule $M_R^{(nr)} \otimes M_L^{(nr)}$ in such a way that $\widetilde{M}_{R(P)}^{(nr)} \otimes \widetilde{M}_L^{(nr)}$ is an extended internal (pseudounramified) bilinear Hilbert semispace $H_a^{+, (nr)}$ according to definitions 1.8.

Then, the i -th class $\{\widetilde{M}_{\overline{v}_i, m_i}^{(nr)} \otimes \widetilde{M}_{v_i, m_i}^{(nr)}\}_{m_i}$ of $\widetilde{M}_R^{(nr)} \otimes \widetilde{M}_L^{(nr)}$ corresponds to the extended internal bilinear Hilbert subsemispace $H_a^{+, (nr)}(i)$; so that we get the towers

$$\begin{aligned}
H_a^{+, nr}(1) &\subset \cdots \subset H_a^{+, nr}(i) \subset \cdots \subset H_a^{+, nr}(q), \\
H_a^+(1) &\subset \cdots \subset H_a^+(i) \subset \cdots \subset H_a^+(q),
\end{aligned}$$

of embedded pseudounramified and pseudoramified bilinear Hilbert subsemispaces.

Taking into account the isomorphism between the algebraic and analytic bilinear Hilbert semispaces H_a^+ and H_h^+ , corresponding towers of embedded analytic bilinear Hilbert subsemispaces can also be envisaged:

$$\begin{aligned}
H_h^{+, nr}(1) &\subset \cdots \subset H_h^{+, nr}(i) \subset \cdots \subset H_h^{+, nr}(q), \\
H_h^+(1) &\subset \cdots \subset H_h^+(i) \subset \cdots \subset H_h^+(q).
\end{aligned}$$

2. Let $\widetilde{M}_{R_{L_\oplus}}^{nr} \otimes_{(D)} \widetilde{M}_{L_\oplus}^{nr} = \bigoplus_{i=1}^q (\widetilde{M}_{\bar{v}_i; m_i}^{nr} \otimes_{(D)} \widetilde{M}_{v_i; m_i}^{nr})$ be the decomposition of the bisemimodule over the $GL_n(L_{\bar{v}_\oplus}^{nr} \times L_{v_\oplus}^{nr})$ -bisemimodule $M_{R_{L_\oplus}}^{nr} \otimes_{(D)} M_{L_\oplus}^{nr}$. Then, the algebraic pseudounramified extended (resp. diagonal) bilinear Hilbert semispace $H_{a_\oplus}^{+,nr}$ (resp. $\mathcal{H}_{a_\oplus}^{+,nr}$) decomposes according to:

$$H_{a_\oplus}^{+,nr} = \bigoplus_{i=1}^q H_a^{+,nr}(i) \quad (\text{resp. } \mathcal{H}_{a_\oplus}^{+,nr} = \bigoplus_{i=1}^q \mathcal{H}_a^{+,nr}(i))$$

where $\widetilde{M}_{\bar{v}_i; m_i}^{nr} \otimes \widetilde{M}_{v_i; m_i}^{nr} \simeq H_a^{+,nr}(i)$.

So, we can construct a **tower of direct sums** of embedded algebraic pseudounramified extended (resp. diagonal) bilinear Hilbert subsemispaces

$$H_a^{+,nr}\{1\} \subset \cdots \subset H_a^{+,nr}\{i\} \subset \cdots \subset H_a^{+,nr}\{q\}$$

such that:

- $H_a^{+,nr}\{q\} \equiv H_{a_\oplus}^{+,nr} = \bigoplus_{\nu=1}^q H_a^{+,nr}(\nu)$,
- $H_a^{+,nr}\{i\} = \bigoplus_{\nu=1}^i H_a^{+,nr}(\nu)$,
- $\mathcal{H}_a^{+,nr}\{i\} = \bigoplus_{\nu=1}^i \mathcal{H}_a^{+,nr}(\nu)$,

refer respectively to the q -th, i -th and i -th **state** of H_a^+ , $H_a^{+,nr}$ and $\mathcal{H}_a^{+,nr}$.

3. Considering the isomorphism between the algebraic and analytic bisemimodules $(M_R^{nr} \otimes_{(D)} M_L^{nr})$ and $(M_R^{s,nr} \otimes_{(D)} M_L^{s,nr})$, a tower of direct sums of embedded analytic pseudounramified extended (resp. diagonal) bilinear Hilbert subsemispaces also exists:

$$H_h^{+,nr}\{1\} \subset \cdots \subset H_h^{+,nr}\{i\} \subset \cdots \subset H_h^{+,nr}\{q\}$$

and

$$\mathcal{H}_h^{+,nr}\{1\} \subset \cdots \subset \mathcal{H}_h^{+,nr}\{i\} \subset \cdots \subset \mathcal{H}_h^{+,nr}\{q\}$$

such that:

- $H_h^{+,nr}\{q\} \equiv H_{h_\oplus}^{+,nr} = \bigoplus_{\nu=1}^q H_h^{+,nr}(\nu)$,

- $H_h^{+,nr}\{i\} = \bigoplus_{\nu=1}^i H_h^{+,nr}(\nu) ,$
- $\mathcal{H}_h^{+,nr}\{i\} = \bigoplus_{\nu=1}^i \mathcal{H}_h^{+,nr}(\nu) ,$

refer respectively to the q -th, i -th and i -th **state** of $H_h^{+,nr}$, $H_h^{+,nr}$ and $\mathcal{H}_h^{+,nr}$.

4. If the decomposition of the pseudoramified bisemisheaf $\widetilde{M}_{R_L} \otimes_{(D)} \widetilde{M}_L$ over the $GL_n(L_{\overline{v}} \times L_v)$ -bisemimodule $M_{R_L} \otimes_{(D)} M_L$ is envisaged, then the algebraic pseudoramified bilinear Hilbert semispaces H_a^+ and \mathcal{H}_a^+ decompose according to:

- $H_a^+\{i\} = \bigoplus_{j=1}^i H_a^+(j) ;$
- $\mathcal{H}_a^+\{i\} = \bigoplus_{j=1}^i \mathcal{H}_a^+(j) ; \quad 1 \leq j \leq i ,$

where $1 \leq i \leq q$. This leads to **towers of direct sums** of embedded algebraic and analytic pseudoramified bilinear Hilbert subsemispaces, i.e. **towers of states of these bilinear Hilbert semispaces** H_a^+ , H_h^+ , \mathcal{H}_a^+ and \mathcal{H}_h^+ :

- $H_a^+\{1\} \subset \cdots \subset H_a^+\{i\} \subset \cdots \subset H_a^+\{q\} ,$
- $H_h^+\{1\} \subset \cdots \subset H_h^+\{i\} \subset \cdots \subset H_h^+\{q\} ,$
- $\mathcal{H}_a^+\{1\} \subset \cdots \subset \mathcal{H}_a^+\{i\} \subset \cdots \subset \mathcal{H}_a^+\{q\} ,$
- $\mathcal{H}_h^+\{1\} \subset \cdots \subset \mathcal{H}_h^+\{i\} \subset \cdots \subset \mathcal{H}_h^+\{q\} ,$

where $H_a^+\{1\} \equiv H_a^+(1)$.

The towers of embedded bilinear Hilbert subsemispaces lead to consider that these bilinear Hilbert semispaces are “solvable” and thus graded.

Definition 3.13 Projectors: (a) Let

$$\begin{aligned} H_a^{+,nr}(1) &\subset \cdots \subset H_q^{+,nr}(i) \subset \cdots \subset H_a^{+,nr}(q) , \\ H_a^+(1) &\subset \cdots \subset H_a^+(i) \subset \cdots \subset H_a^+(q) \end{aligned}$$

be the two towers of embedded pseudounramified and pseudoramified bilinear Hilbert subsemispaces introduced in section 3.12.

Then, the following projectors:

$$\begin{aligned} P_{i_{R \times L}}^{\text{fac}(nr)} : H_a^{+,nr}(q) &\longrightarrow H_a^{+,nr}(i), \quad \forall 1 \leq i \leq q, \\ P_{i_{R \times L}}^{\text{fac}} : H_a^+(q) &\longrightarrow H_a^+(i), \end{aligned}$$

can be introduced, as it is done classically, in such a way that:

- $P_{i_{R \times L}}^{\text{fac}(nr)}$ projects $H_a^{+,nr}(q)$ onto the i -th pseudounramified bilinear Hilbert subsemispace $H_a^{+,nr}(i)$;
- $P_{i_{R \times L}}^{\text{fac}}$ projects $H_a^+(q)$ onto the i -th pseudoramified bilinear Hilbert subsemispace $H_a^+(i)$.

(b) Let $H_{a_{\oplus}}^+$ be an extended bilinear Hilbert semispace decomposing according to:

$$\bullet H_{a_{\oplus}}^+ = \bigoplus_{i=1}^q H_a^+\{i\} \text{ such that } H_a^+\{i\} = \bigoplus_{\nu=1}^i H_a^+(\nu) ;$$

$$\text{or } \bullet H_{a_{\oplus}}^{+,nr} = \bigoplus_{i=1}^q H_a^{+,nr}\{i\} \text{ such that } H_a^{+,nr}\{i\} = \bigoplus_{j=1}^i H_a^{+,nr}(j) .$$

Then, we can define the (bi)projectors of states:

$$\begin{aligned} P_{i_{R \times L}}^{nr} : H_{a_{\oplus}}^{+,nr} &\longrightarrow H_a^{+,nr}\{i\}, & 1 \leq i \leq q, \\ P_{i_{R \times L}} : H_{a_{\oplus}}^+ &\longrightarrow H_a^+\{i\}, \end{aligned}$$

mapping $H_{a_{\oplus}}^{+,nr}$ respectively into its closed extended bilinear subsemispace $H_a^{+,nr}\{i\}$ which is the i -th (bisemi)state.

The (bi)projectors $P_{i_{R \times L}}^{nr}$ and $P_{i_{R \times L}}$ are idempotent (bi)operators in such a way that the mappings they generate are inverse deformations (of Galois representations) as proved by the author elsewhere [31].

Proposition 3.14 *The operator $T_{R,L}^D(\text{Pra } \Gamma_{R,L}^{[m]})$ (resp. $T_{R,L}^D(\Gamma_{R,L}^{[m]})$) is a random operator decomposing into a set of operators $\{T_{R,L}^D(g_{R,L}^{[m]}(i))\}_i$ (resp. $\{T_{R,L}^D(\gamma_{R,L}^{h[m]}(i))\}_i$), $\forall 1 \leq i \leq q$ according to the shifted pseudoramified (resp. pseudounramified) conjugacy classes of $GL_{n[m]}(L_v^{nr} \otimes \mathbb{R})$ associated with the $T_m^{(t)}(\mathbb{R})$ -principal bundle.*

Proof: Indeed, according to section 3.5, a random operator $T_{R,L}^D(\text{Pra } \Gamma_{R,L}^{[m]}) = \{T_{R,L}^D(g_{R,L}^{[m]}(i))\}_{i=1}^q$ (resp. $T_{R,L}^D(\Gamma_{R,L}^{[m]}) = \{T_{R,L}^D(\gamma_{R,L}^{[m]}(i))\}_{i=1}^q$), acting on an extended bilinear Hilbert semispace $H_a^{\mp(nr)}$, is a set $\{T_{R,L}^D(g_{R,L}^{[m]}(i))\}_{i=1}^q \in \{\mathbb{M}_{R,L}(H_a^{\mp}(i))\}$ (resp. $\{T_{R,L}^D(\gamma_{R,L}^{[m]}(i))\}_{i=1}^q \in \{\mathbb{M}_{R,L}(H_a^{\mp(nr)}(i))\}$) such that the bilinear form:

$$\begin{aligned} t_R(\ell, m) &= (T_R^D(g_R^{[m]}(\ell)) e_R^\ell, e_L^m) \\ (\text{resp. } t_R(\mu, \nu) &= (T_R^D(\gamma_R^{[m]}(\mu)) e_R^\mu, e_L^\nu)) \end{aligned}$$

or

$$\begin{aligned} t_L(\ell, m) &= (e_R^\ell, T_L^D(g_L^{[m]}(m)) e_L^m) \\ (\text{resp. } t_L(\mu, \nu) &= (e_R^\mu, T_L^D(\gamma_L^{[m]}(\nu)) e_L^\nu)) \end{aligned}$$

be measurable.

$\mathbb{M}_{R,L}(H_a^{\mp}(i))$ (resp. $\mathbb{M}_{R,L}(H_a^{\mp(nr)}(i))$) is a von Neumann subsemialgebra relative to bounded operators on a closed connected subsemispace $H_a^{\mp(nr)}(i)$ of $H_a^{\pm(nr)}$ referring to the i -th conjugacy class of $GL_n(L_{\bar{v}}^{(nr)} \times L_v^{(nr)})$.

These considerations are made in complete analogy with what is known for random operators on linear Hilbert (semi)spaces [9]. ■

Proposition 3.15 1) Let $T_{R,L}^D(g_{R,L}^{[m]}(u))$ and $T_{R,L}^D(g_{R,L}^{[m]}(v))$ be two right or left random operators such that $u < v$. Then, the random operator $T_{R,L}^D(g_{R,L}^{[m]}(v))$ is an “extension” of the random operator $T_{R,L}^D(g_{R,L}^{[m]}(u))$ corresponding to a difference of conjugacy classes $(v - u)$.

2) Let $T_{R,L}^D(\gamma_{R,L}^{[m]}(o))$ and $T_{R,L}^D(\gamma_{R,L}^{[m]}(p))$ be two right or left random operators such that $o < p$. Then, $T_{R,L}^D(\gamma_{R,L}^{[m]}(p))$ is an “extension” of $T_{R,L}^D(\gamma_{R,L}^{[m]}(o))$ corresponding to a difference of conjugacy classes $(p - o)$.

Definition 3.16 Towers of pseudoramified and pseudounramified von Neumann subsemialgebras: (a) In connection with the definition 3.13 introducing towers of direct sums of embedded bilinear Hilbert subsemispaces, we shall define here towers of sums of random operators:

$$\begin{aligned} T_{R,L}^D(g_{R,L}^{[m]} \{i\}) &= \bigoplus_{j=1}^i T_{R,L}^D(g_{R,L}^{[m]}(j)) \\ (\text{resp. } T_{R,L}^D(\gamma_{R,L}^{[m]} \{i\}) &= \bigoplus_{j=1}^i T_{R,L}^D(\gamma_{R,L}^{[m]}(j))), \end{aligned}$$

such that

$$\begin{aligned} T_{R,L}^D(g_{R,L}^{[m]}\{i\}) &\in \mathbb{M}_{R,L}(H_a^\mp\{i\}) , \quad 1 \leq i \leq q , \\ (\text{resp. } T_{R,L}^D(\gamma_{R,L}^{[m]}\{i\}) &\in \mathbb{M}_{R,L}(H_a^{\mp,nr}\{i\}) , \end{aligned}$$

where $\mathbb{M}_{R,L}(H_a^\mp\{i\})$ (resp. $\mathbb{M}_{R,L}(H_a^{\mp,nr}\{i\})$) is the pseudoramified (resp. pseudounramified) von Neumann subsemialgebra of the i -th state referring to the i -th sum of random operators.

So, a tower of pseudoramified and pseudounramified von Neumann subsemialgebras of states can be introduced by:

$$\begin{aligned} \mathbb{M}_{R,L}(H_a^\mp\{1\}) &\subset \cdots \subset \mathbb{M}_{R,L}(H_a^\mp\{i\}) \subset \cdots \subset \mathbb{M}_{R,L}(H_a^\mp\{q\}) , \\ (\text{resp. } \mathbb{M}_{R,L}(H_a^{\mp,nr}\{1\}) &\subset \cdots \subset \mathbb{M}_{R,L}(H_a^{\mp,nr}\{i\}) \subset \cdots \subset \mathbb{M}_{R,L}(H_a^{\mp,nr}\{q\}) , \end{aligned}$$

such that

$$\begin{aligned} \mathbb{M}_{R,L}(H_a^\mp\{i\}) &= \bigoplus_{j=1}^i \mathbb{M}_{R,L}(H_a^\mp(j)) \\ (\text{resp. } \mathbb{M}_{R,L}(H_a^{\mp,nr}\{i\}) &= \bigoplus_{j=1}^i \mathbb{M}_{R,L}(H_a^{\mp,nr}(j)) . \end{aligned}$$

(b) Similarly, on the towers

$$\begin{aligned} H_h^{\mp,nr}(1) &\subset \cdots \subset H_h^{\mp,nr}(i) \subset \cdots \subset H_h^{\mp,nr}(q) \\ \text{and } H_h^\mp(1) &\subset \cdots \subset H_h^\mp(i) \subset \cdots \subset H_h^\mp(q) \end{aligned}$$

of analytic Hilbert subspaces introduced in definition 3.12, the corresponding towers of pseudounramified and pseudoramified von Neumann subsemialgebras will be given by:

$$\mathbb{M}_{R,L}(H_h^{\mp,nr}(1)) \subset \cdots \subset \mathbb{M}_{R,L}(H_h^{\mp,nr}(i)) \subset \cdots \subset \mathbb{M}_{R,L}(H_h^{\mp,nr}(q)) , \quad 1 \leq i \leq q \leq \infty ,$$

and by

$$\mathbb{M}_{R,L}(H_h^\mp(1)) \subset \cdots \subset \mathbb{M}_{R,L}(H_h^\mp(i)) \subset \cdots \subset \mathbb{M}_{R,L}(H_h^\mp(q)) .$$

Proposition 3.17 *Let $\mathbb{M}_{R,L}(H_a^{\mp,(nr)})$ be the von Neumann semialgebra of bounded self-adjoint operators on the smooth extended bilinear Hilbert semispace $H_a^{\mp,(nr)}$.*

*Let $\mathbb{M}_{R,L}(H_a^{\mp,(nr)}\{i\})$ be the von Neumann subsemialgebra of random operators on the closed smooth **extended** bilinear subsemispace $H_a^{\mp,nr}\{i\}$ and let $\mathbb{M}_{R,L}(\mathcal{H}_a^{\mp,nr}\{i\})$ be the corresponding von Neumann subsemialgebra on the closed smooth **internal** diagonal bilinear subsemispace $\mathcal{H}_a^{\mp,nr}\{i\}$.*

Then, the discrete spectrum $\sigma(T_{R,L}^D)$ of an operator $T_{R,L}^D \in \mathbb{M}_{R,L}(H_a^{\mp,(nr)})$ is obtained by the morphism:

$$\begin{array}{ccc} i_{\{i\}_{R,L}}^a \circ i_{\{i\}_{R,L}}^a : & \mathbb{M}_{R,L}(H_a^{\mp,(nr)}) & \longrightarrow [\mathbb{M}_{R,L}(\mathcal{H}_a^{\mp,(nr)}\{i\})]_i, \\ & T_{R,L}^D & \longrightarrow \sigma(T_{R,L}^D) \end{array}$$

where $i_{\{i\}_{R,L}}^a$ and $i_{\{i\}_{R,L}}^a$ are given by:

$$\begin{array}{ccc} i_{\{i\}_{R,L}}^a : & \mathbb{M}_{R,L}(H_a^{\mp,(nr)}) & \longrightarrow [\mathbb{M}_{R,L}(H_a^{\mp,(nr)}\{i\})]_i, \\ i_{\{i\}_{R,L}}^a : & [\mathbb{M}_{R,L}(H_a^{\mp,(nr)}\{i\})]_i & \longrightarrow [\mathbb{M}_{R,L}(\mathcal{H}_a^{\mp,(nr)}\{i\})]_i. \end{array}$$

Proof: First remark that $\mathbb{M}_{R,L}(H_a^{\mp,(nr)})$ is a non-abelian von Neumann semialgebra since the extended bilinear Hilbert semispace $H_a^{\mp,(nr)}$ constitutes the enveloping (semi)algebra of the semimodule $M_L^{(nr)}$ (resp. $M_R^{(nr)}$).

The morphism

$$\begin{array}{ccc} i_{\{i\}_{R,L}}^a : & \mathbb{M}_{R,L}(H_a^{\mp,(nr)}) & \longrightarrow [\mathbb{M}_{R,L}(H_a^{\mp,(nr)}\{i\})]_i, \\ & T_{R,L}^D(\text{Pra } \Gamma_{R,L}^{[m]}) & \longrightarrow [T_{R,L}^D(g_{R,L}^{[m]}\{i\})]_i \\ \text{(resp. } & T_{R,L}^D(\Gamma_{R,L}^{[m]}) & \longrightarrow [T_{R,L}^D(\gamma_{R,L}^{[m]}\{i\})]_i), \end{array}$$

transforms the bounded operator $T_{R,L}^D(\text{Pra } \Gamma_{R,L}^{[m]})$ (resp. $T_{R,L}^D(\Gamma_{R,L}^{[m]})$) into the set $[T_{R,L}^D(g_{R,L}^{[m]}\{i\})]_i$ (resp. $[T_{R,L}^D(\gamma_{R,L}^{[m]}\{i\})]_i$) of bounded operators (i.e. sums of random operators acting on closed subsemispace $M_{L,R,L}\{i\}$ whose sums of enveloping subsemispace are $H_h^{\mp,(nr)}\{i\}$).

On the other hand, the isomorphism $i_{\{i\}_{R,L}}^a$

$$\begin{array}{ccc} i_{\{i\}_{R,L}}^a : & [\mathbb{M}_{R,L}(H_a^{\mp,(nr)}\{i\})]_i & \longrightarrow [\mathbb{M}_{R,L}(\mathcal{H}_a^{\mp,(nr)}\{i\})]_i, \\ & [T_{R,L}^D(g_{R,L}^{[m]}\{i\})]_i & \longrightarrow \sigma(T_{R,L}^D), \\ \text{(resp. } & [T_{R,L}^D(\gamma_{R,L}^{[m]}\{i\})]_i & \longrightarrow \sigma_{nr}(T_{R,L}^D), \end{array}$$

transforms the non-abelian von Neumann subsemialgebra $\mathbb{M}_{R,L}(H_a^{\mp,(nr)}\{i\})$ into the abelian or diagonal von Neumann subsemialgebra $\mathbb{M}_{R,L}(\mathcal{H}_a^{\mp,(nr)}\{i\})$ of sums of random operators. $[\mathbb{M}_{R,L}(\mathcal{H}_a^{\mp,(nr)}\{i\})]_i$ is then an algebra of the sum of random operators acting on diagonal enveloping subsemialgebras $(\mathcal{H}_a^{\mp,(nr)}\{i\})$. $\sigma(T_{R,L}^D)$ (resp. $\sigma_{nr}(T_{R,L}^D)$) is thus the pseudo-ramified (resp. pseudounramified) spectrum of the bounded operator $T_{R,L}^D$. ■

Corollary 3.18 *Let $\mathbb{M}_{R \times L}(H_a^{\mp, (nr)})$ be the von Neumann bisemialgebra of bounded bioperators $T_R^D \otimes T_L^D$ on $H_a^{\mp, (nr)}$ and let $\mathbb{M}_{R \times L}(H_a^{\mp, (nr)}\{i\})$ be the i -th corresponding von Neumann subbisemialgebra of the sum of random bounded bioperators on $H_a^{\mp, (nr)}\{i\}$. If $\mathbb{M}_{R \times L}^h(\mathcal{H}_a^{\mp}\{i\})$ is the i -th von Neumann diagonal subbisemialgebra of random diagonal bioperators $T_R^D\{i\} \otimes_D T_L^D\{i\}$ on $\mathcal{H}_a^{\mp, nr}\{i\}$, then the discrete spectrum $\sigma(T_R^D \otimes T_L^D)$ of $(T_R^D \otimes T_L^D) \in \mathbb{M}_{R \times L}(H_a^{\mp, (nr)})$ is obtained by the morphism:*

$$\begin{aligned} i_{\{i\}_{R \times L}}^a \circ i_{\{i\}_{R \times L}}^a : \quad \mathbb{M}_{R \times L}(H_a^{\mp, (nr)}) &\longrightarrow [\mathbb{M}_{R \times L}(\mathcal{H}_a^{\mp, (nr)}\{i\})]_i, \\ T_{R \times L}^D &\longrightarrow \sigma(T_{R \times L}^D), \end{aligned}$$

where $T_{R \times L}^D$ is the condensed notation for $T_R^D \otimes T_L^D$.

$\mathbb{M}_{R, L}(H_a^{\pm, (nr)})$ then corresponds to a solvable (bi)semialgebra.

Proof: This corollary is an extension of the preceding proposition to the bioperator $(T_R^D(\Gamma_R) \otimes T_L^D(\Gamma_L))$. ■

3.19. Shifted global pseudounramified (resp. pseudoramified) elliptic bisemi-modules: Referring to proposition 3.6, the action of the differential bioperator $(T_R^D \otimes T_L^D)$ on the bisemisheaf $(\widetilde{M}_R^{(nr)} \otimes \widetilde{M}_L^{(nr)})$ over the $GL_n(L_{\overline{v}}^{(nr)} \times L_v^{(nr)})$ -bisemimodule $(M_R^{(nr)} \otimes M_L^{(nr)})$ consists in mapping it into the shifted bisemisheaf $(\widetilde{M}_{R_{n[m]}}^{(nr)} \otimes \widetilde{M}_{L_{n[m]}}^{(nr)})$ over the $GL_{n[m]}((L_{\overline{v}}^{(nr)} \otimes \mathbb{R}) \times (L_v^{(nr)} \otimes \mathbb{R}))$ -bisemimodule $(M_{R_{n[m]}}^{(nr)} \otimes M_{L_{n[m]}}^{(nr)})$ such that $(\widetilde{M}_{R_{n[m] \oplus}}^{(nr)} \otimes \widetilde{M}_{L_{n[m] \oplus}}^{(nr)})$ decomposes into “ q ” subbisemisheaves.

But, according to proposition 2.10 referring to the Langlands global program introduced in [29], there is a bijection between the $GL_n(L_{\overline{v} \oplus}^{(nr)} \times L_{v \oplus}^{(nr)})$ -bisemimodule $(M_{R \oplus}^{(nr)} \otimes M_{L \oplus}^{(nr)})$ and its cuspidal counterpart given by the global pseudoramified (resp. pseudounramified) elliptic $G_{s_{R \times L}}$ -bisemimodule:

$$\begin{aligned} \text{ELLIP}_{R \times L}(n, q) &= \sum_{i=1}^q \sum_{m_i} \lambda(n, i, m_i) e^{-2\pi i(i)z} \otimes \sum_{i=1}^q \sum_{m_i} \lambda(n, i, m_i) e^{2\pi i(i)z}, \quad z \in \mathbb{R}^n, \\ (\text{resp. } \text{ELLIP}_{R \times L}^{nr}(n, q) &= \sum_{i=1}^q \sum_{m_i} \lambda_{nr}(n, i, m_i) e^{-2\pi i(i)z} \otimes \sum_{i=1}^q \sum_{m_i} \lambda_{nr}(n, i, m_i) e^{2\pi i(i)z}), \end{aligned}$$

such that we have the commutative diagram:

$$\begin{array}{ccc}
\widetilde{M}_R^{nr} \otimes \widetilde{M}_L^{nr} & \xrightarrow{T_R^D \otimes T_L^D} & \widetilde{M}_{R_{n[m]}}^{nr} \otimes \widetilde{M}_{L_{n[m]}}^{nr} \\
\downarrow & & \downarrow \\
\text{ELLIP}_{R \times L}^{nr}(n, q) & \xrightarrow{T_R^D \otimes T_L^D} & \text{ELLIP}_{R \times L}^{nr}(n[m], q) \\
\downarrow & & \downarrow \\
\text{ELLIP}_{R \times L}(n, q) & \xrightarrow{T_R^D \otimes T_L^D} & \text{ELLIP}_{R \times L}(n[m], q) \\
\uparrow & & \uparrow \\
\widetilde{M}_R \otimes \widetilde{M}_L & \xrightarrow{T_R^D \otimes T_L^D} & \widetilde{M}_{R_{n[m]}} \otimes M_{L_{n[m]}}
\end{array}$$

where $\text{ELLIP}_{R \times L}^{nr}(n[m], q)$ (resp. $\text{ELLIP}_{R \times L}^{nr}(n[m], q)$) is the shifted global pseudounramified (resp. pseudoramified) elliptic $((G_{s_R} \otimes \mathbb{R}) \times (G_{s_L} \otimes \mathbb{R})$ -bisemimodule.

As an application of proposition 3.17, we suggest the following proposition [32].

Proposition 3.20 *The shifted global pseudounramified (resp. pseudoramified) n -dimensional elliptic bisemimodule*

$$\begin{aligned}
& \text{ELLIP}_{R \times L}^{nr}(n[m], q) = \text{ELLIP}_R^{nr}(n[m], q) \otimes \text{ELLIP}_L^{nr}(n[m], q) \\
& (\text{resp. } \text{ELLIP}_{R \times L}(n[m], q) = \text{ELLIP}_R(n[m], q) \otimes \text{ELLIP}_L(n[m], q)),
\end{aligned}$$

gives rise to (or is functorially equivalent to) the eigenbivalue equation of the i -th (bi)states:

$$\begin{aligned}
& (T_R^D \otimes T_L^D)(\text{ELLIP}_R^{nr}(n, i) \otimes \text{ELLIP}_L^{nr}(n, i)) \\
& = E_R^{nr}\{n, i\} \times E_L^{nr}\{n, i\} \cdot (\text{ELLIP}_R^{nr}(n, i) \otimes \text{ELLIP}_L^{nr}(n, i)), \quad 1 \leq i \leq q, \\
& (\text{resp. } (T_R^D \otimes T_L^D)(\text{ELLIP}_{R \times L}(n, i)) \\
& = E_R\{n, i\} \times E_L(\{n, i\} \cdot (\text{ELLIP}_{R \times L}(n, i)), \quad 1 \leq i \leq q.
\end{aligned}$$

Proof:

1. The shifted global pseudounramified elliptic bisemimodule $\text{ELLIP}_{R \times L}^{(nr)}(n[m], q)$ generates the eigenbivalue equation:

$$\text{ELLIP}_{R \times L}^{nr}(n[m], i) = (E_{R \times L}^{nr}\{n, i\})(\text{ELLIP}_{R \times L}^{nr}(n, i))$$

which can be rewritten according to [32]:

$$(T_R^D \otimes T_L^D)(\text{ELLIP}_{R \times L}^{nr}(n, i)) = (E_R^{nr}\{n, i\} \times E_L^{nr}\{n, i\})(\text{ELLIP}_{R \times L}^{nr}(n, i))$$

where the right (resp. left) eigenvalue $E_R^{nr}(\{n, i\})$ (resp. $E_L^{nr}(\{n, i\})$) corresponds to a sum over the i first pseudounramified algebraic classes of shifts into m dimensions of the Hecke characters $\lambda_{nr}(n, \nu, m_\nu)$ (resp. $\lambda_{nr}(n, \nu, m_\nu)$) i.e. to infinitesimal generators of the considered Lie algebra, $1 \leq \nu \leq i$.

2. The bisemialgebra of von Neumann $\mathbb{M}_{R \times L}(H_h^{\mp, nr})$ can then be considered as a solvable bisemialgebra generating a tower of sums of pseudounramified von Neumann subbisemialgebras according to definition 3.16. On the other hand, the set of pseudounramified eigenbivalues of $(T_R^D \otimes T_L^D)$ forms an embedded sequence:

$$E_R^{nr}\{n, 1\} \cdot E_L^{nr}\{n, 1\} \subset \cdots E_R^{nr}\{n, i\} \cdot E_L^{nr}\{n, i\} \subset \cdots E_R^{nr}\{n, q\} \cdot E_L^{nr}\{n, q\}$$

in one-to-one correspondence with the set of embedded eigenbifunctions given by the product, right by left, of the truncated Fourier series at “ i ” terms:

$$\text{ELLIP}_{R \times L}^{nr}(n, i) = \sum_{\nu=1}^i \sum_{m_\nu} \lambda_{nr}(n, \nu, m_\nu) e^{-2\pi i \nu z} \otimes \sum_{\nu=1}^i \sum_{m_\nu} \lambda_{nr}(n, \nu, m_\nu) e^{2\pi i \nu z}, \quad z \in \mathbb{R}^n.$$

3. The proof was given for the “pseudounramified” case, taking into account that the “pseudoramified” case can be handled similarly. \blacksquare

Proposition 3.21 *The discrete spectrum $\sigma(T_{R,L}^D)$ of $(T_{R,L}^D) \in \mathbb{M}_{R \times L, R \times L}(H_h^{\mp, (nr)})$ and the discrete spectrum $\sigma^a(T_{R,L}^D)$ of $(T_{R,L}^D) \in \mathbb{M}_{R \times L, R \times L}(H_a^{\mp, (nr)})$ are isomorphic (and often equal).*

Proof: Consider the commutative diagram:

$$\begin{array}{ccc} \mathbb{M}_{R \times L, R \times L}(H_h^{\mp, (nr)}) & \xleftarrow{i_{\mathbb{M}}^a{}_{R,L} - \mathbb{M}^h{}_{R,L}} & \mathbb{M}_{R \times L, R \times L}(H_a^{\mp, (nr)}) \\ \downarrow i_{\{i\}}^h{}_{R,L} & & \downarrow i_{\{i\}}^a{}_{R,L} \\ [\mathbb{M}_{R \times L, R \times L}(H_h^{\mp, (nr)}\{i\})]_i & \xleftarrow{i_{R,L}^{a-h}} & [\mathbb{M}_{R \times L, R \times L}(H_a^{\mp, (nr)}\{i\})]_i \\ \downarrow i_{\{i\}}^h{}_{R,L}^D & & \downarrow i_{\{i\}}^a{}_{R,L}^D \\ [\mathbb{M}_{R \times L, R \times L}(\mathcal{H}_h^{\mp, (nr)}\{i\})]_i & \xrightarrow{i_D^{a-h}} & [\mathbb{M}_{R \times L, R \times L}(\mathcal{H}_a^{\mp, (nr)}\{i\})]_i \end{array}$$

where

- the isomorphism $i_{\mathbb{M}}^a{}_{R,L} - \mathbb{M}^h{}_{R,L}$ has been introduced in proposition 3.3;

- the morphisms $i_{\{i\}}^h$ and $i_{\{i\}}^a$ result from the decomposition of $\mathbb{M}_{R \times L}^{R,L}(H_h^{\mp, (nr)})$ and of $\mathbb{M}_{R \times L}^{R,L}(H_a^{\mp, (nr)})$ into sums of pseudounramified or pseudoramified subbisemialgebras (see definition 3.16).

From the isomorphism $i_{D_{R,L}}^{a-h}$, it results that the discrete spectrum $\sigma(T_{R,L}^D)$ of $T_{R,L}^D \in \mathbb{M}_{R \times L}^{R,L}(H_h^{\mp, (nr)})$ and the discrete spectrum $\sigma^a(T_{R,L}^D)$ of $T_{R,L}^D \in \mathbb{M}_{R \times L}^{R,L}(H_a^{\mp, (nr)})$ are isomorphic. So, we get the thesis. ■

3.22. Factors of von Neumann:

- We are now interested in the classification of the factors of von Neumann, i.e. in von Neumann algebras having trivial centers (reduced to \mathbb{C}). According to definition 3.16, we see that two types of towers of von Neumann sub(bi)semialgebras have been introduced:
 - the first referring to **pseudounramified (algebraic) classes** of the bilinear Hilbert semispaces $H_a^{\mp, nr}$ (or $H_h^{\mp, nr}$) on which they have been defined;
 - the second referring to **pseudoramified (algebraic) classes** of the bilinear Hilbert semispaces H_a^{\mp} (or H_h^{\mp}).

So, the classification of factors of von Neumann will be based on these two types of towers of von Neumann subsemialgebras on bilinear Hilbert (sub)semispaces which are associated with Hecke sublattices as developed in proposition 2.3 (proof). As a result, the dimensions of the factors of von Neumann will directly refer to Hecke sublattices.

- The bilinear Hilbert semispaces H_a^{\mp} , isomorphic to H_h^{\mp} , constituting the natural representation spaces of the von Neumann (bi)semialgebras, were supposed to be pseudoramified in the sense that the $GL_n(L_{\bar{v}} \times L_v)$ -bisemimodule $(M_{R_L} \otimes M_L)$ is pseudoramified. That is to say that the $T_n(L_{v_i})$ -subsemimodule M_{v_i} (as well as $M_{\bar{v}_i}$) has a rank given by $n_i = i^n \cdot N^n$ (see section 1.5).

On the other hand, the corresponding pseudounramified $T_n(L_{v_i}^{nr})$ -subsemimodule $M_{v_i}^{nr}$ would have a rank $n_i^{nr} = i^n$ according to [29], which allows to envisage the introduction of pseudounramified bilinear Hilbert subsemispaces, noted $H_a^{nr}(i)$, as it was defined in section 3.12.

Proposition 3.23 (Classification of (bi)factors of von Neumann with respect to algebraic dimensions)

1. **Type I_i** : on the pseudounramified bilinear Hilbert semispace H_a^{nr} , there are q factors $\mathbb{M}_{R,L}(H_a^{nr}(i))$ of type I_i , $1 \leq i \leq q \leq \infty$, where i denotes a global residue degree.
2. **Type II_1** : on the bilinear Hilbert subsemispace $H_a^{\mp, \text{in}}[L_{\overline{v}^1} \times L_{v^1}]$ restricted to the representation space of the bilinear parabolic subsemigroup $P_n(L_{\overline{v}^1} \times L_{v^1})$, there are N subfactors $\mathbb{M}_{R,L}(H_a^{\mp, \text{in}}(i))$, $1 \leq i \leq N$, of type II_{1_i} , where i denotes an internal algebraic dimension corresponding to the number of automorphisms of the global inertia subgroup.
The factor $\mathbb{M}_{R,L}(H_a^{\mp, \text{in}}(N))$ is the factor of type II_1 .
3. **Type $II_{(\infty)}$** : on the tensor products $H_a^{\mp}(i) = H_a^{nr}(i) \otimes H_a^{\mp, \text{in}}(N)$ of the pseudounramified bilinear Hilbert semispace $H_a^{nr}(i)$ by the bilinear Hilbert subsemispace $H_a^{\mp, \text{in}}(N)$, there are q pseudoramified factors $\mathbb{M}_{R,L}(H_a^{nr}(i) \otimes H_a^{\mp}(N))$ of type $II_{(\infty)}$, $1 \leq i \leq q \leq \infty$, where i denotes a global residue degree.
4. **Type II_{∞}** : on the tensor products $H_a^{nr}(\infty) \otimes H_a^{\mp, \text{in}}(j)$, $1 \leq j \leq N$, the factors $\mathbb{M}_{R,L}(H_a^{nr}(\infty)) \otimes \mathbb{M}_{R,L}(H_a^{\mp, \text{in}}(j))$, of type II_{∞} are defined.

Proof:

1. As there are q conjugacy classes of the pseudounramified bilinear Hilbert semispace H_a^{nr} , there are q “pseudounramified” factors $\mathbb{M}_{R,L}(H_a^{nr}(i))$ in the tower:

$$\mathbb{M}_{R,L}(H_a^{nr}(1)) \subset \cdots \subset \mathbb{M}_{R,L}(H_a^{nr}(i)) \subset \cdots \subset \mathbb{M}_{R,L}(H_a^{nr}(q))$$

as introduced in sections 3.12, 3.13 and 3.16.

So, there are q factors of type I_i , $1 \leq i \leq q \leq \infty$ with minimal projections.

2. If we consider the N internal conjugacy classes of the bilinear parabolic semigroup $P_n(L_{\overline{v}^1} \times L_{v^1})$ corresponding to the (shifted) intermediate inner automorphisms of the global inertia subgroups $I_{L_{v_i}}$ having an order N , we can introduce on $H_a^{\mp}[L_{\overline{v}^1} \times L_{v^1}]$ a tower of inner hyperfinite subfactors [23], [24]:

$$\mathbb{M}_{R,L}(H_a^{\mp, \text{in}}(1)) \subset \cdots \subset \mathbb{M}_{R,L}(H_a^{\mp, \text{in}}(i)) \subset \cdots \subset \mathbb{M}_{R,L}(H_a^{\mp, \text{in}}(N))$$

in such a way that:

- the index $[\mathbb{M}_{R,L}(H_a^{\mp,\text{in}}(i)) : \mathbb{M}_{R,L}(H_a^{\mp,\text{in}}(1))] = i$ of the i -th hyperfinite subfactor with respect to the first hyperfinite subfactor is the internal algebraic dimension (see section 3.7).
- the upper hyperfinite subfactor $\mathbb{M}_{R,L}(H_a^{\mp,\text{in}}(N))$ is the hyperfinite factor of type II_1 having an index N and corresponding to the order of the global inertia subgroup $I_{L_{v_i}}$.

Indeed, if we take into account proposition 2.9, the Hecke characters on sublattices associated with $\mathbb{M}_{R,L}(H_a^{\mp,\text{in}}(i))$, $1 \leq i \leq N$, must take values in the interval $[0, 1]$: they then correspond to the continued dimensions [17] of the classes of the projectors of the subfactors of type II_1 of von Neumann algebras on a linear Hilbert semispace.

3. As on the pseudounramified bilinear Hilbert subsemispaces $H_a^{nr}(i)$, pseudounramified factors $\mathbb{M}_{R,L}(H_a^{nr}(i))$ of type I_i are defined and as, on the bilinear Hilbert subsemispace $H_a^{\mp,\text{in}}(N)$, a factor of type II_1 is defined, it is evident that, on their tensor products $H_a^{nr}(i) \otimes H_a^{\mp,\text{in}}(N)$, pseudoramified factors of type II_i , characterized by minimal projections, $1 \leq i \leq q \leq \infty$, can be defined, the factor of type II_1 “ramifying” the pseudounramified factors I_i .
4. And, then, the classical factors of Araki-Woods [1], [15] of type II_∞ correspond to the factors $\mathbb{M}_{R,L}(H_a^{nr}(i = \infty)) \otimes \mathbb{M}_{R,L}(H_a^{\mp,\text{in}}(j))$, $1 \leq j \leq N$, where
 - $\mathbb{M}_{R,L}(H_a^{nr}(i = \infty))$ is the pseudounramified factor of type I_∞ ;
 - $\mathbb{M}_{R,L}(H_a^{\mp,\text{in}}(j))$ is the hyperfinite subfactor of type II_{1_j} . ■

Corollary 3.24 *The equivalent of a factor of type III_λ can be obtained by considering the cross product of the factor II_∞ by a subgroup of automorphisms of it [1], [15].*

Proof: Indeed, a factor M_λ of type M_λ [28] is isomorphic to the cross product of a factor “ N ” of type II_∞ by Aut “ N ” [14], [16], [33]. ■

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